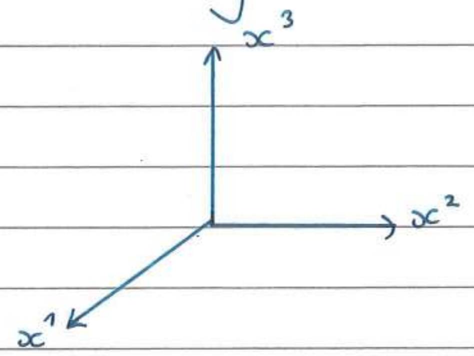
General Relativity: Lecture 1

1.1 - Newtonian theory

Physical laws on $\mathbb{R}^3 \times \mathbb{R}^1$ with universal time t

Inertial frame: coordinates $\{x^1, x^2, x^3\}$ for non-accelerating observer.



Transformation between inertial frames

$$x^i \mapsto x'^i = R^i{}_j x^j - v^i t + a^i$$

$R^i{}_j$: rotation $SO(3)$
v^i : relative velocity
a^i : spatial translation

$$t \mapsto t' = t + b$$

b : time translation

Principle of Relativity: laws of physics independent of choice of inertial frame.

Gravity :- gravitational field affects motion of particles

- field generated by a mass distribution.

Gravitational potential

$$\Phi : \mathbb{R}^3 \to \mathbb{R}$$
$$x^i \mapsto \Phi(x)$$

obeys Poisson equation and produces acceleration

$$\nabla^2 \Phi = 4\pi G \rho \,, \quad m \frac{d^2}{dt^2} \underline{x} = m \ddot{\underline{x}} = - m \underline{\nabla} \Phi$$

where $\rho : \mathbb{R}^3 \to \mathbb{R}$ is matter density

G : Newton's constant, "strength of gravity"

∇^2 : Laplacian on $\mathbb{R}^3$, $\delta^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}$

Example : localised spherical mass

$$M = \int_0^R d^3x \, \rho$$

$$\Phi = - \frac{GM}{|\underline{x}|}$$

$$\Rightarrow \quad \ddot{x}^i = - \frac{GM}{|\underline{x}|^2} \frac{\underline{x}}{|\underline{x}|}$$

1.2 - Problems with Newtonian theory

a) Instantaneous signal propagation

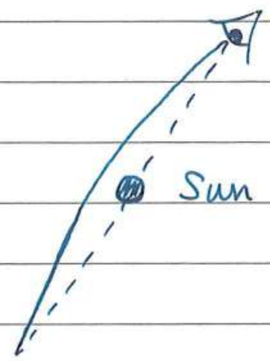
No time derivatives in Poisson equation
⇒ instant signalling

Incompatible with Special Relativity (signals limited by c)

b) Bending of light

If light is a wave, no coupling between electromagnetism and gravity, so no deflection.

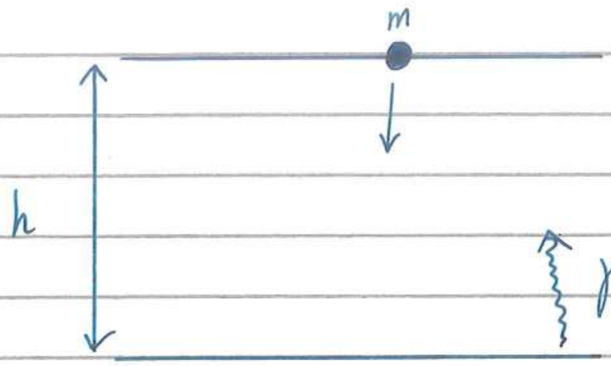
If light is a particle (massless test particle), get 1/2 of deflection observed



Eddington 1919

Wait for solar eclipse then compare location of stars near sun.

c) Thought experiment (Einstein)



- Drop mass m from height h
- A bottom, convert m to energy as photon
- Photon measured again at top.

$$E_m^{top} = mc^2$$

$$E_m^{bottom} = mc^2 + \frac{1}{2}mv^2 + O(v^4)$$

$$= mc^2\left(1 + \frac{gh}{c^2} + \dots\right)$$

↑ SR corrections.

$g \simeq 10\ ms^{-2}$, grav. acc. at Earth's surface.

$\frac{gh}{c^2} \ll 1$ for small SR effects. (linear terms)

$$E_m^{bottom} = E_\gamma^{bottom} \simeq mc^2(1 + gh/c^2)$$

Conservation of energy $\Rightarrow E_\gamma^{top} = mc^2$

But $\dfrac{E_\gamma^{top}}{E_\gamma^{bottom}} \simeq \dfrac{1}{1 + gh/c^2} \simeq 1 - \dfrac{gh}{c^2}$

So energy (and frequency) of photon must decrease on way back up - "<u>gravitational redshift</u>".

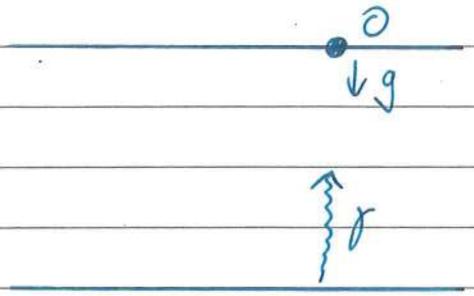
Clock at bottom runs slow! "Time dilation"

Frame at rest is <u>not</u> inertial

No mechanism for this to happen in Newtonian gravity!

- gravity couples to <u>mass</u>, not energy.
- energy conserved, earth ensures momentum conserved.
- redshift observed by Pound-Rebka 1960

Consider an observer $\mathcal{O}$ falling freely from to



$t = 0$: photon released from bottom, $\mathcal{O}$ at rest

$t = h/c$: photon reaches top, $\mathcal{O}$ has speed $u = gh/c$

There will be a SR velocity redshift for $\mathcal{O}$

$$E_\gamma^{\mathcal{O}} = \left(\frac{1 + u/c}{1 - u/c}\right)^{1/2} E_\gamma^{top} \simeq (1 + u/c)\, E_\gamma^{top}$$

$$\simeq \left(1 + \frac{gh}{c^2}\right) E_\gamma^{top}$$

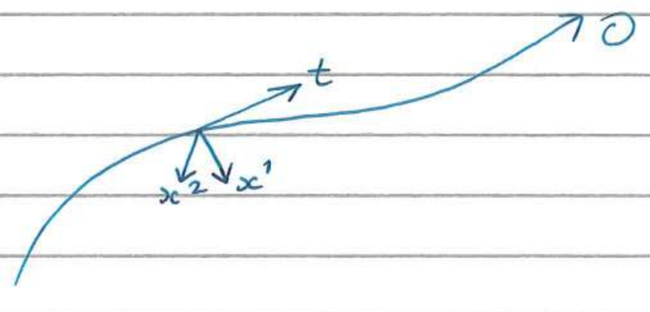
So $$\frac{E_\gamma^{\mathcal{O}}}{E_\gamma^{bottom}} = \frac{E_\gamma^{\mathcal{O}}}{E_\gamma^{t}} \frac{E_\gamma^{t}}{E_\gamma^{b}} \simeq \frac{1 + gh/c^2}{1 + gh/c^2} \simeq 1 + \mathcal{O}(g^2h^2)$$

The redshift of the photon <u>vanishes</u> in the freely falling reference frame.

<u>Equivalence Principle</u>: may eliminate effects of gravity locally by moving to a freely falling reference frame

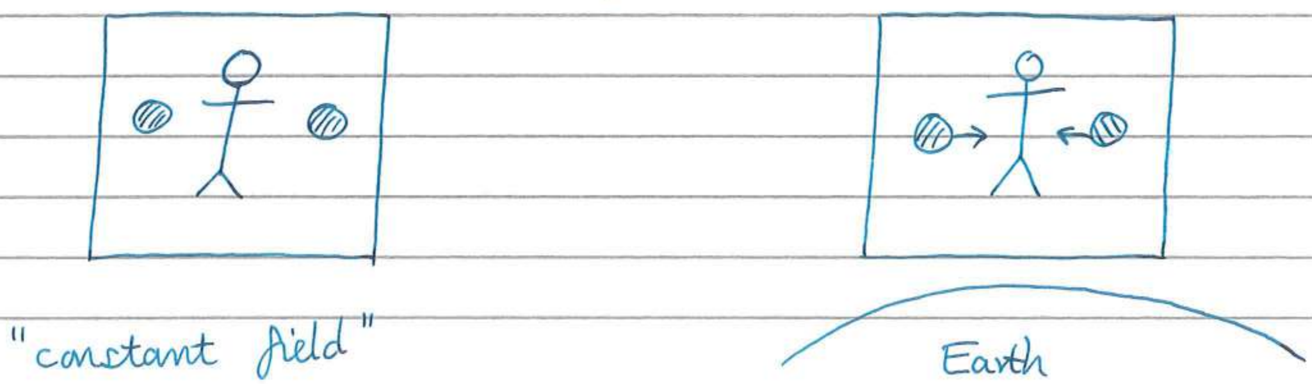
i.e. local effects of gravity indistinguishable from being in an accelerated frame of reference.

A freely falling observer $\mathcal{O}$ may use a <u>local</u> inertial frame $\{x^\mu\}$ in a small neighbourhood of a point on their worldline



Special Relativity holds in this local frame.

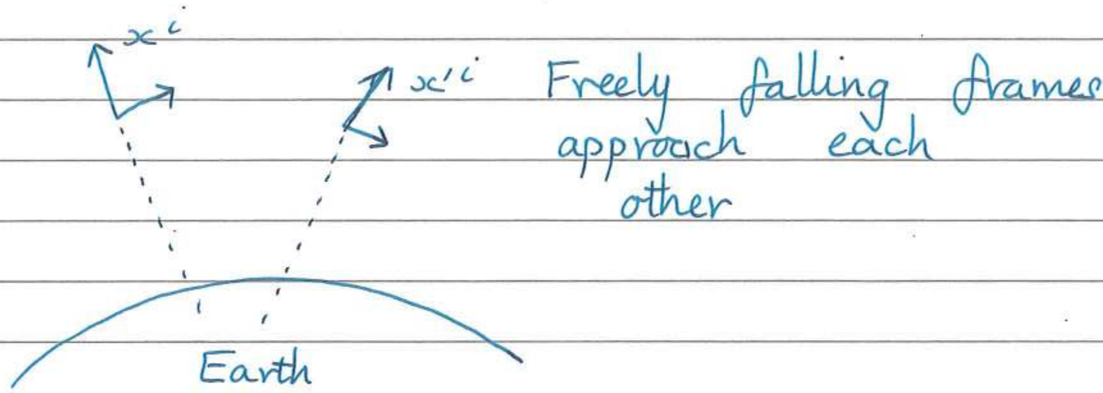
<u>Example</u>: elevator falling to earth



If elevator is small and you only consider a short time, these look the same!

<u>Local</u>: small neighbourhood in space and time.

Gravity appears as <u>relative acceleration</u> of local inertial frames



Transformation between these frames is not a (linear) Lorentz transformation

$$x'^a \neq L^a{}_b x'^b$$

$\Rightarrow$ Spacetime is <u>curved</u>.

Need a new theory :- SR holds over short distances / times

- Equations will be non-linear (energy sources field)

- Reduces to Newtonian gravity for weak fields and low speeds.

2. Review of Special Relativity

Model spacetime on $\mathbb{R}^4$

- Flat "Minkowski" space M_4.
- Points in M_4 are "events"
- Label points with $x^\mu = (t, x^i) = (x^0, x^i)$ (but no invariant meaning)

Conventions: use "summation convention"

- Inertial coordinates $(\mu, \nu, \ldots)$
- General coordinates $(a, b, \ldots)$
- Euclidean coordinates $(i, j, \ldots)$

M_4 has natural measure of distance

$$ds^2 = -(dx^0)^2 + \sum_i (dx^i)^2$$

$$= \eta_{\mu\nu}\, dx^\mu dx^\nu$$

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$$

Coordinates in which this holds are <u>inertial frames</u>

Lorentz transformations

Linear transformations of x^μ that leave ds^2 invariant

$$x^\mu \mapsto x'^\mu = L^\mu{}_\nu x^\nu$$

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \equiv \eta_{\mu\nu} dx'^\mu dx'^\nu$$

$$\Rightarrow \eta_{\mu\nu} L^\mu{}_\sigma L^\nu{}_\rho = \eta_{\sigma\rho}$$

In matrix notation

$$x' = L x \, , \qquad L^T \eta L = \eta$$

These transformations relate "inertial frames" in which speed of light is the same.
Their defining property implies

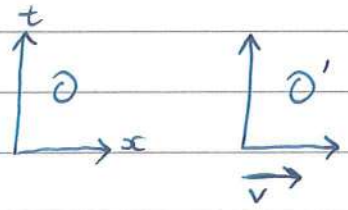
- $\det L = \pm 1$
- $L \in O(1,3)$

Minkowski version of $R^T \mathbb{1} R = \mathbb{1}$ for $O(4)$

Elements can be broken into "boosts" and "rotations"

<u>Boosts</u> :- "rotations" which mix space and time

- relate frames moving relative to each other

e.g. 

$$x'^{\mu} = L^{\mu}{}_{\nu} x^{\nu}$$

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\gamma = (1 - v^2)^{-1/2}$$

<u>Rotations</u> : Only in $\mathbb{R}^3 \subset \mathbb{R}^{1,3}$, $x'^{i} = R^{i}{}_{j} x^{j}$

$$L^{\mu}{}_{\nu} = \begin{pmatrix} 1 & \\ & R^{i}{}_{j} \end{pmatrix}, \quad R \in SO(3)$$

<u>Discrete symmetries</u> : "Parity" $x^i \mapsto -x^i$

"Time reversal" $x^0 \to -x^0$

Generally, can also shift the coordinates so that

$$x'^{\mu} = L^{\mu}{}_{\nu} x^{\nu} + c^{\mu}$$

constant

Together, these give <u>Poincaré transformations</u>.

Causal structure

Metric η defines <u>Lorentz invariant</u> distance between events P and Q, with coordinates x_P^μ and x_Q^μ

$$(\Delta x)^2 = \eta_{\mu\nu}(x^\mu_P - x^\mu_Q)(x^\nu_P - x^\nu_Q)$$

$$= \eta(x_P - x_Q, x_P - x_Q)$$

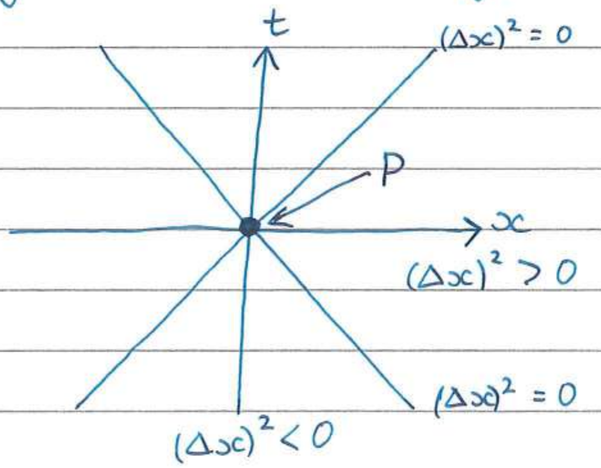
Depending on the sign of $(\Delta x)^2$, the events P and Q are called

$(\Delta x)^2 < 0$ "<u>timelike separated</u>"

$(\Delta x)^2 = 0$ "<u>lightlike separated</u>"

$(\Delta x)^2 > 0$ "<u>spacelike separated</u>"

The set of events that are lightlike separated from P define the "<u>lightcone</u>" at P

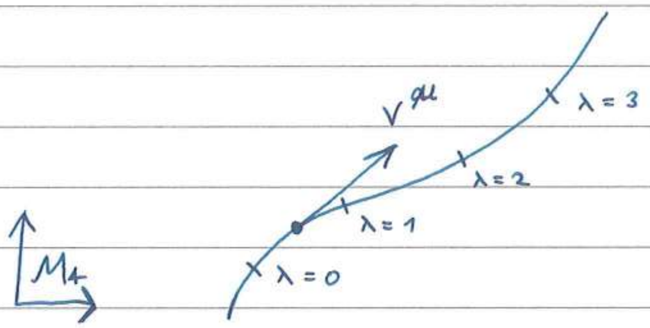


Future lightcone

Past lightcone

Curves and tangent vectors

Curve given by a map $\lambda \mapsto x^\mu(\lambda)$ $(\mathbb{R} \to M_4)$



Tangent vector to curve at $x^\mu(\lambda_0)$ is

$$v^\mu = \frac{d}{d\lambda} x^\mu(\lambda) \Big|_{\lambda = \lambda_0}$$

The tangent vector v^μ is called

$\eta(v, v) < 0$ "timelike"

$\eta(v, v) = 0$ "null"

$\eta(v, v) > 0$ "spacelike"

The sign of $\eta(v,v)$ depends on the image of the curve, not its parametrisation.

A curve whose tangent vector is everywhere timelike is a "timelike curve", etc.

- Massive particles follow timelike curves
- Massless particles follow null curves.

Proper time

A natural parametrisation for a timelike curve is given by the Lorentz invariant "proper time" τ along the curve.

$$x^\mu = x^\mu(\tau)$$

$$d\tau = \sqrt{-ds^2}$$

$$= \sqrt{-\eta_{\mu\nu}\, dx^\mu\, dx^\nu}$$

$$= \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}\; d\tau$$

$$\Rightarrow \; -\eta_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu = 1 \qquad \left(\dot{x}^\mu := \frac{dx^\mu}{d\tau} \right)$$

Similarly, spacelike curves parametrised by "proper distance" ds.

As τ is Lorentz invariant

$$\dot{x}'^\mu(\tau) = \frac{d}{d\tau} x'^\mu(\tau) = \frac{\partial x'^\mu}{\partial x^\nu} \frac{dx^\nu}{d\tau} = L^\mu{}_\nu\, \dot{x}^\nu(\tau)$$

Lorentz vectors (4-vectors)

Objects with components V^μ that transform under Lorentz transformations as

$$V^\mu \mapsto V'^\mu = L^\mu{}_\nu V^\nu$$

The metric $\eta_{\mu\nu}$ gives an indefinite scalar product on vectors.

The scalar product $\eta_{\mu\nu} V^\mu W^\nu$ is a Lorentz scalar, so is invariant under Lorentz transformations

$$\eta_{\mu\nu} V'^\mu W'^\nu = \eta_{\mu\nu} L^\mu{}_\rho V^\rho L^\nu{}_\sigma W^\sigma$$

$L^T \eta L = \eta$

$$= \eta_{\mu\nu} V^\mu W^\nu$$

A vector is timelike, null, ... depending on sign of $\eta(V,V)$.

Other Lorentz tensors

Scalars are invariant, e.g. $\eta(V,W)$

Co-vectors (or 1-forms) transform under dual representation.

$$\Lambda = (L^T)^{-1} = \eta L \eta^{-1}$$

$$u'_\mu = \Lambda_\mu{}^\nu u_\nu \,, \qquad \Lambda_\mu{}^\nu = \eta_{\mu\rho} L^\rho{}_\sigma \eta^{\sigma\nu}$$
$$= u_\nu (L^{-1})^\nu{}_\mu$$

Vectors are elements of $T_p \mathcal{M}_4$. (at a point P)

Covectors are elements of $T_p^* \mathcal{M}_4$

Covectors define a map $T_p \mathcal{M}_4 \longrightarrow \mathbb{R}$

$$u: v \in T_p \mathcal{M}_4 \mapsto u(v) = u_\mu v^\mu \in \mathbb{R}.$$

The metric $\eta_{\mu\nu}$ defines an isomorphism

$$T_p \mathcal{M}_4 \cong T_p^* \mathcal{M}_4$$

$$v^\mu = \eta^{\mu\nu} v_\nu \,, \qquad v_\mu = \eta_{\mu\nu} v^\nu$$

(p,q) tensors transform like a product of vectors and covectors

$$T^{\mu_1 .. \mu_p}{}_{\nu_1 .. \nu_q} \mapsto T'^{\mu_1 .. \mu_p}{}_{\nu_1 .. \nu_q}$$

$$= L^{\mu_1}{}_{\rho_1} \dots L^{\mu_p}{}_{\rho_p} (L^{-1})^{\sigma_1}{}_{\nu_1} \dots (L^{-1})^{\sigma_q}{}_{\nu_q} T^{\rho_1 .. \rho_p}{}_{\sigma_1 .. \sigma_q}$$

Products e.g. $v^\mu w^\nu u_\rho$ are $(2,1)$ tensor

- Linear combinations of (p,q) tensors are (p,q).

- Products and contractions of tensors are again tensors.

- Type of tensors can be read off from indices.

Tensor fields

Assignment of a tensor to each point of $\mathcal{M}_4$

$$T : x \mapsto T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q}(x)$$

e.g. given $V^\mu(x)$, $\eta_{\mu\nu} V^\mu(x) V^\nu(x) = \eta(V,V)(x)$ is a scalar function.

e.g. given scalar function $f(x)$

$$u_\mu(x) := \frac{\partial}{\partial x^\mu} f(x) \equiv \partial_\mu f$$

is a covector / 1-form.

e.g. Wave operator $\Box = \eta^{\mu\nu} \partial_\mu \partial_\nu$ maps (p,q) to (p,q)

$(\nabla^2 = \Box)$

e.g. $\eta_{\mu\nu}$ is a $(0,2)$ tensor that does not depend on x or change with Lorentz transformations

i.e. $\partial_\mu \eta_{\nu\rho} = 0$, $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$

e.g. Identity tensor / Kronecker delta

$$\delta^\mu{}_\nu = \mathrm{diag}(1,1,1,1)$$ in any inertial frame.

$$\delta^\mu{}_\nu T^{\nu\cdots}{}_{\cdots} = T^{\mu\cdots}{}_{\cdots}$$

$$\eta^{\mu\rho}\eta_{\nu\rho} = \delta^\mu{}_\nu$$

e.g. $T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$

$T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$

Worldlines of massive particles

Timelike curve parametrised by proper time τ

$$u^\mu = \dot{x}^\mu(\tau)$$ "4-velocity"

$$u^\mu u_\mu = \eta_{\mu\nu} u^\mu u^\nu = -1$$

e.g. $u^\mu = (1, \vec{0})$ at rest

$u^\mu = \gamma(v)(1, \vec{v})$ massive

$u^\mu = (1, \vec{v})$, $|\vec{v}|^2 = 1$ massless

4-acceleration : $$a^\mu = \frac{d}{d\tau} u^\mu$$

Massive free particles obey $a^\mu = 0$

Note that $\frac{d}{d\tau}\left(u^\mu u^\nu \eta_{\mu\nu} \right) = 0$

$$= 2\, a^\mu u^\nu \eta_{\mu\nu}$$

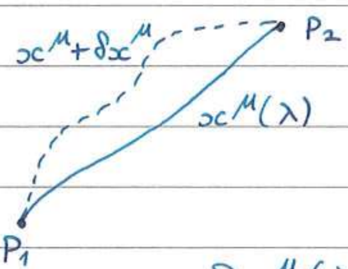
$$\Rightarrow \quad \eta(a, u) = 0$$

If $a^\mu \neq 0$, a^μ orthogonal to u^μ

$$\Rightarrow \quad a_\mu a^\mu > 0 \quad \text{"spacelike"}$$

Action and proper time

Physical trajectories extremise proper time



$$\delta x^\mu(\lambda_1) = \delta x^\mu(\lambda_2) = 0$$

Action for free particle

$$S[x] = -m \int d\tau$$

$$= -m \int \sqrt{-\eta_{\mu\nu}\, dx^\mu dx^\nu}$$

With a parametrisation $x^\mu(\lambda)$

$$S[x] = \int d\lambda \, \mathcal{L}_\lambda$$

$$\mathcal{L}_\lambda = -m\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$

e.g. $\lambda = x^0 = t$

$$\mathcal{L}_t = -m\sqrt{1-|\vec{v}|^2}$$

$$\vec{v} = \frac{d\vec{x}}{dt}$$

Can always choose λ s.t.

$$\left(-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2} = \text{constant}$$

λ is then an "affine parameter"
$\lambda = a\tau + b$ for τ proper time.

<u>Important</u>: for a timelike curve $x^\mu(\lambda)$, extremising $S[x]$ is equivalent to extremising

$$S[x] = \int d\lambda \, \frac{1}{2}\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

This simpler functional can also be used for <u>massless</u> particles!

Curves / trajectories / worldlines that extremise the action are known as "<u>geodesics</u>".

- Analogue of straight lines in M_4.

4 - momentum

Defined as $p^\mu = m u^\mu = (E, \vec{p})$

$$p_\mu p^\mu = -m^2 \quad \text{"mass shell relation"}$$

rest mass

Given a reference frame / coordinates in which

- particle has momentum p^μ
- observer $\mathcal{O}$ has velocity v^μ

Observer will <u>measure</u> energy of particle to be

$$E_{\mathcal{O}} = -\eta_{\mu\nu} p^\mu v^\nu$$

e.g. Particle at rest w.r.t. $\mathcal{O}$

- $p^\mu = m v^\mu$

$$E_{\mathcal{O}} = m \quad \text{"rest mass in } \mathcal{O} \text{ frame"}$$

Can work in frame of $\mathcal{O}$ $\left(p^\mu = (m, \vec{0}),\ v^\mu = (1, 0) \right)$
or in a different frame $\left(v^\mu = \gamma(1, \vec{v}),\ p^\mu = m v^\mu \right)$.

e.g. Particle moving at v in $\hat{x}$ direction in $\bar{\mathcal{O}}$ frame

$$v^\mu = (1, \vec{0}) \quad \text{"}\mathcal{O} \text{ at rest in own frame"}$$

$$p^\mu = \gamma m(1, v, 0, 0)$$

$$\Rightarrow E_{\bar{\mathcal{O}}} = m\gamma \cong \underbrace{m}_{\text{rest mass}} + \underbrace{\tfrac{1}{2}v^2}_{\text{kinetic energy}} + \ldots$$

Stress - energy tensor

Energy density, energy flux, momentum density and pressure encoded by a symmetric $(2,0)$ tensor $T^{\mu\nu}$

Let $T^{\mu\nu}$ be stress - tensor and u^μ the 4 - velocity of a timelike observer $\mathcal{O}$ in an inertial frame

- $\mathcal{O}$ measures a 4 - momentum density

$$j^\mu = -T^{\mu\nu} u_\nu$$

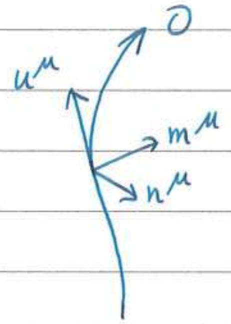
- Energy is $-u^\mu p_\mu$, so energy density is

$$\rho = -u^\mu j_\mu = T_{\mu\nu} u^\mu u^\nu$$

- for normal matter, $T_{\mu\nu} u^\mu u^\nu \geq 0$, "<u>weak energy condition</u>".

Let m^μ and n^μ be spacelike vectors s.t.

- $m_\mu m^\mu = n_\mu n^\mu = +1$
- $u_\mu m^\mu = u_\mu n^\mu = 0$

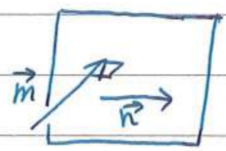


<u>Pressure</u> in m^μ direction measured by O is

$$p = T_{\mu\nu} m^\mu m^\nu$$

<u>Stress</u> in n^μ direction across surface $\perp$ to m^μ is

$$S = T_{\mu\nu} m^\mu n^\nu$$



<u>Conservation</u> of energy and momentum is

$$\partial_\mu T^{\mu\nu} = 0$$

- Can derive this from Noether's theorem for translations
- For a field theory coupled to gravity

with action $S[\phi, g_{ab}]$

$$T_{ab} = \frac{\delta S}{\delta g^{ab}}$$

Perfect fluid

Macroscopic description of matter in terms of pressure and density.

e.g. gases or fluids.

A "perfect fluid" is one such that an observer in the rest frame of the fluid sees it as isotropic

- "comoving" observer sees it as rotation invariant.

In reference frame of fluid

$$T_{00} = \rho \, , \quad T_{ij} = p\, \delta_{ij} \, , \quad T_{0i} = 0$$

- ρ : "energy density".
- p : "pressure".

To specify the kind of fluid, give a relation between ρ and p

"<u>Equation of state</u>": $p = p(\rho)$

In an inertial frame where fluid moves with 4-velocity u^μ

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p \eta_{\mu\nu}$$

e.g. for $u^\mu = (1, \vec{0})$, $T_{\mu\nu} u^\mu u^\nu = \rho$

"<u>Dust</u>" is pressureless ($p = 0$, particles do not bump into each other much) - project along u^μ for energy conservation.

$$u_\mu \partial_\nu T^{\nu\mu} = u_\mu \partial_\nu (\rho u^\nu u^\mu)$$

$$= u_\mu u^\mu \underbrace{\partial_\nu (\rho u^\nu)}_{j^\nu} + \rho u^\nu \underbrace{u_\mu \partial_\nu u^\mu}_{0 \text{ as } \partial_\nu (u_\mu u^\mu) = 0}$$

$$= -\partial_\nu j^\nu$$

"Fluid density conserved".

In $|v| \ll 1$ limit, $u^\mu = \gamma(1, \vec{v}) \simeq (1, \vec{v})$

$$j^\mu = \rho u^\mu \simeq (\rho, \rho\vec{v}),$$

$$\partial_\mu j^\mu = \partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \quad \text{"conservation of mass"}$$

"continuity equation"

4.4 - Electromagnetic flux

Electromagnetic field encoded in a $(0,2)$ anti-symmetric tensor F_{ab}

- $F_{\mu\nu}$ has 6 components (3 electric, 3 magnetic)

An observer with 4-velocity v^μ measures

- Electric field : $E_\mu = F_{\mu\nu} v^\nu$

  E^μ is spacelike with 3 independent components (see PS1)

- Magnetic field : $B_\mu = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\nu\rho} v^\sigma$

  where $\epsilon_{0123} = +1$ and $\epsilon_{\mu\nu\rho\sigma} = \epsilon_{[\mu\nu\rho\sigma]}$

  Again, B^μ spacelike with 3 components.

Example : $\mathcal{O}$ at rest, $v^\mu = (1, \underline{0})$

$$E_\mu = F_{\mu 0} \quad \Rightarrow \quad E_i = F_{i0}$$

$$B_\mu = -\frac{1}{2} \epsilon_{\mu\nu\rho 0} F^{\nu\rho}$$

$$= +\frac{1}{2} \epsilon_{0\mu\nu\rho} F^{\nu\rho}$$

$$\Rightarrow B_i = \frac{1}{2} \epsilon_{ijk} F^{jk}$$

$$\Rightarrow \quad F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

Example: $\mathcal{O}$ with $v^\mu = \gamma(v)(1, \underline{v})$

$$E'_i = F_{i\mu} v^\mu$$

$$= \gamma(v)\left(F_{i0} + F_{ij} v^j\right)$$

$$= \gamma(v)\left(E_i + \tfrac{1}{2}\epsilon_{ijk} v^j B^k\right)$$

$$\Rightarrow \quad \vec{E}' = \gamma(v)\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

Electric vs. magnetic is dependent on the inertial frame.

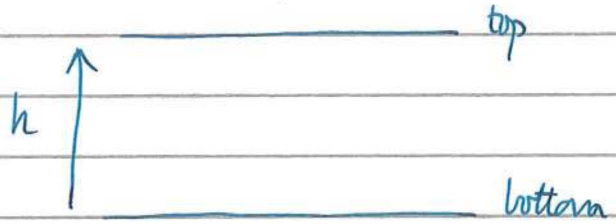
5.1 - Review

In absence of gravity, spacetime is

- $M_4 = \mathbb{R}^4$ as topological space
- Flat metric of signature $(-,+,+,+)$
- There are distinguished inertial frames in which $g = \eta = \mathrm{diag}(-1,1,1,1)$

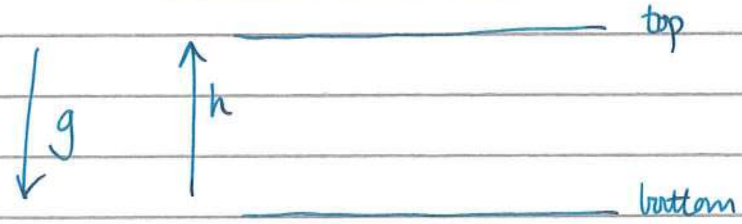
5.2 - Gravity (intuition)

Observers at rest in a <u>global</u> inertial frame agree on time differences



$$\Delta t_t = \Delta t_b$$

Observers at rest in a gravitational field experience <u>time dilation</u>



$$\frac{\Delta t_b}{\Delta t_t} \simeq 1 - gh$$

Can eliminate local effects by moving to a freely falling frame.

5.3 - Gravity (concretely)

There are no global inertial frames in a (non-uniform) gravitational field.

A freely falling observer may set up a local inertial frame in which $g = \eta$

Spacetime is a smooth, 4-dimensional manifold with a metric g_{ab} of signature $(-,+,+,+)$

5.4 - Smooth manifolds

set of points with neighbourhoods a satisfying some axioms

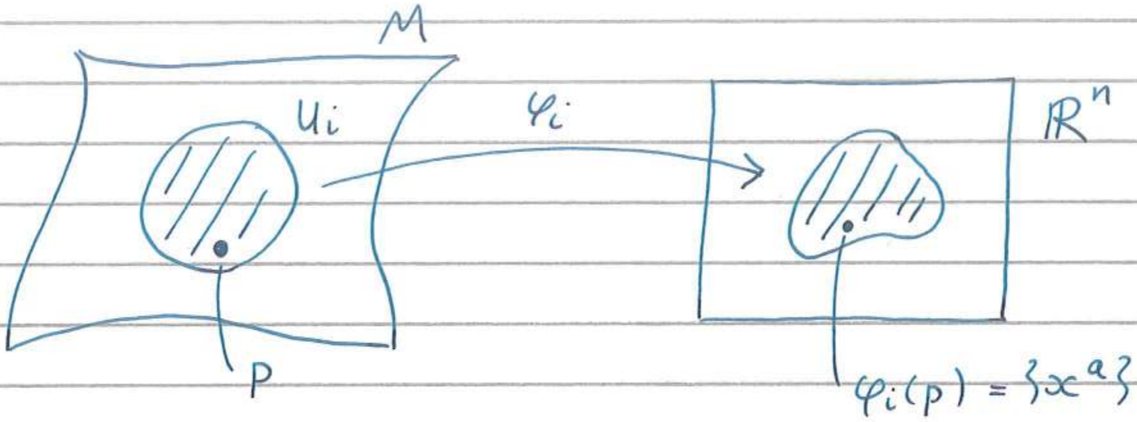
Def: 1) M is a topological space

2) M has coordinate charts $\{U_i, \varphi_i\}$

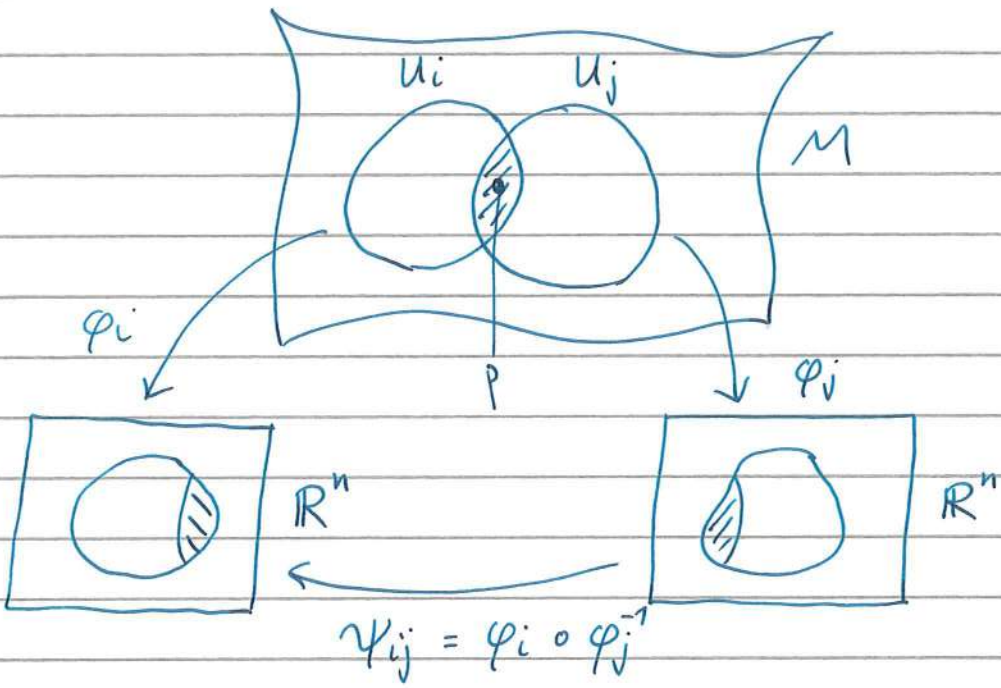
labels the chart

- $U_i \subset M$ are open sets such that $\bigcup_i U_i = M$

- $\varphi_i : U_i \to \mathbb{R}^n$, homeomorphism
  - bijection
  - continuous
  - continuous inverse



3) Compatibility on overlaps



If $\varphi_i(p) = \{x^a\}$

$p \in U_i \cap U_j$

$\varphi_j(p) = \{x'^a\}$

$x'^a = x'^a(x)$ are C^∞

These are <u>general</u> coordinate transformations!

- Might need multiple sets of coordinates / patches to cover manifold.
- If patches fully overlap, just a redefinition of your coordinates

5.5 - Tensor fields

On overlap $U_i \cap U_j$ s.t. $\varphi_i(p) = \{x^a\}$
$\varphi_j(p) = \{x'^a\}$

Example: scalar $f : M \to \mathbb{R}$

$$f(x) \mapsto f(x') = f(x'(x))$$

Example: tangent vector to curve $x^a(\lambda)$

$$T^a(\lambda) := \frac{dx^a}{d\lambda} \mapsto \frac{dx'^a}{d\lambda} = \frac{\partial x'^a}{\partial x^b}\frac{dx^b}{d\lambda}$$

$$= \frac{\partial x'^a}{\partial x^b} T^b(\lambda)$$

Example: gradient of scalar f

$$\partial_a f = \frac{\partial}{\partial x^a} f \mapsto \frac{\partial}{\partial x'^a} f = \frac{\partial x^b}{\partial x'^a}\frac{\partial}{\partial x^b} f$$

$$= \frac{\partial x^b}{\partial x'^a} \partial_b f$$

Alternatively: tensor C^∞ linear in arguments.
$\omega(V) := \omega_a V^a$, $\omega(fV) = f\,\omega(V)$

Important: $\dfrac{\partial x'^a}{\partial x^b}$ not constant in general!

Example: (p,q) tensor $T^{a_1..a_p}{}_{b_1..b_q}$

$$T^{a_1..a_p}{}_{b_1..b_p} \mapsto T'^{a_1..a_p}{}_{b_1..b_p}(x')$$

$$= \frac{\partial x'^{a_1}}{\partial x^{c_1}} \cdots \frac{\partial x^{d_1}}{\partial x'^{b_1}} \cdots T^{c_1\ldots}{}_{d_1\ldots}(x)$$

As before, we have

- Sum of (p,q) tensors is (p,q)
- Product of (p,q) and (r,s) is $(p+r, q+s)$
- Contraction, $T^{a_1..a_{p-1}c}{}_{b_1..b_{q-1}c}$ is $(p-1, q-1)$

5.6 - Covariant derivatives

Partial derivatives of a tensor do not give another tensor (except for scalars!)

Example: partial derivative of a vector field

$$\frac{\partial V'^a}{\partial x'^b} = \frac{\partial x^c}{\partial x'^b}\frac{\partial}{\partial x^c}\left(\frac{\partial x'^a}{\partial x^d} V^d\right)$$

$$= \underbrace{\frac{\partial x^c}{\partial x'^b} \frac{\partial x'^a}{\partial x^d} \frac{\partial V^d}{\partial x^c}}_{\text{looks right for (1,1) tensor}} + \frac{\partial x^c}{\partial x'^b} \underbrace{\frac{\partial^2 x'^a}{\partial x^c \partial x^d}}_{\text{Non-zero unless } x'^a = L^a{}_b x^b} V^d$$

constant!

Now define <u>covariant derivative</u>

$$D_b V^a = \partial_b V^a + \Gamma^a{}_{bc} V^c$$

- Want $D_b V^a$ to transform as <u>(1,1) tensor</u>.
- $\partial_b V^a$ has extra term so is not a tensor.
- $\Gamma^a{}_{bc}$ defined so that it <u>cancels</u> this extra piece!

$$\Gamma'^a{}_{bc} = \frac{\partial x^p}{\partial x'^b} \frac{\partial x^q}{\partial x'^c} \left(\frac{\partial x'^a}{\partial x^r} \Gamma^r{}_{pq} - \frac{\partial^2 x'^a}{\partial x^p \partial x^q} \right)$$

<u>Exercise</u> on PS2 - check this!

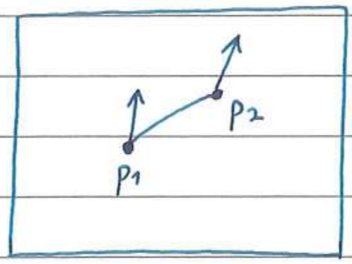
- Check $D_b V^a$ then transforms as (1,1) tensor

$$D_b V^a \mapsto D'_b V'^a = \frac{\partial x^c}{\partial x'^b} \frac{\partial x'^a}{\partial x^d} D_c V^d$$

What is <u>really</u> going on?

Derivative lets you compare a tensor at <u>different</u> points on a manifold

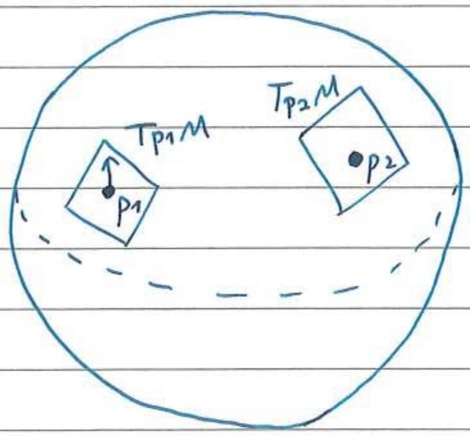
e.g. For $\mathbb{R}^2$



<u>Canonical</u> way to transport vector at p_1 to p_2 to compare (just use ∂_a). $T_{p_1}\mathbb{R}^2 = T_{p_2}\mathbb{R}^2$

Not so simple for curved manifolds.

e.g. for S^2



Prescription for transporting vector from p_1 to p_2 needs <u>extra</u> information.

$T_{p_1}S^2 \cong T_{p_2}S^2$ but no <u>canonical</u> identification

Extra input is a connection ∇_a - allows us to compare vectors in different <u>tangent</u> <u>spaces</u>.

Covariant derivative ∇_a extends to (p,q) tensors as

- Linearity $\nabla_a (S^{b_1 \dots}{}_{c_1 \dots} + T^{b_1 \dots}{}_{c_1 \dots})$

$$= \nabla_a S^{b_1 \dots}{}_{c_1 \dots} + \nabla_a T^{b_1 \dots}{}_{c_1 \dots}$$

- Products $\nabla_a (S^{b_1 \dots}{}_{c_1 \dots} T^{d_1 \dots}{}_{e_1 \dots})$

$$= (\nabla_a S^{b_1 \dots}{}_{c_1 \dots}) T^{d_1 \dots}{}_{e_1 \dots} + S^{b_1 \dots}{}_{c_1 \dots} (\nabla_a T^{d_1 \dots}{}_{e_1 \dots})$$

- Action on scalars $\nabla_a f = \partial_a f$

Example : $\phi = v^a \omega_a$

$$\nabla_a \phi := \partial_a \phi = (\partial_a v^b) \omega_b + v^b (\partial_a \omega_b)$$

but $= \nabla_a (v^b \omega_b)$

$$= (\nabla_a v^b) \omega_b + v^b (\nabla_a \omega_b)$$

$$= (\partial_a v^b + \Gamma^b{}_{ac} v^c) \omega_b + v^b (\nabla_a \omega_b)$$

Holds for any v^b, so

$$\nabla_a \omega_b = \partial_a \omega_b - \Gamma^c{}_{ab} \omega_c$$

Can find action on (p,q) tensor by induction

$$D_a T^{b_1 \dots b_p}{}_{c_1 \dots c_p} = \partial_a T^{b_1 \dots}{}_{c_1 \dots}$$
$$+ \Gamma^{b_1}{}_{ab} T^{b b_2 \dots b_p}{}_{c_1 \dots} + \Gamma^{b_2}{}_{ab} T^{b_1 b b_3 \dots}{}_{c_1 \dots}$$
$$- \Gamma^{c}{}_{ac_1} T^{b_1 \dots}{}_{c c_2 \dots} - \Gamma^{c}{}_{ac_2} T^{b_1 \dots}{}_{c_1 c c_3 \dots}$$

Can think of $\Gamma^c{}_{ab}$ as $(\Gamma_a)^c{}_b$

- For each vector index, $(\Gamma_a \cdot V)^c = (\Gamma_a)^c{}_b V^b$
  $= \Gamma^c{}_{ab} V^b$
- For each 1-form index, $(\Gamma_a \cdot \omega)_b = -(\Gamma_a)^c{}_b \omega_c$
  $= -\Gamma^c{}_{ab} \omega_c$

$\Gamma^a{}_{bc}$ (and D_a) are <u>not</u> unique for a given manifold

$$\Gamma^a{}_{bc} \mapsto \hat{\Gamma}^a{}_{bc} = \Gamma^a{}_{bc} + Q^a{}_{bc}$$

is equally fine if $Q^a{}_{bc}$ is an honest $(1,2)$ tensor

- Have connections that differ by $Q^a{}_{bc}$.
- How to fix this ambiguity?

Physics: Spacetime is a manifold + a metric

- Want to measure distances
- Want ∇_a and metric to be compatible

Peak ahead: **Unique** ∇_a such that

1) $\nabla_a g_{bc} = 0$
2) $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$

This is the connection that appears in GR.

6.1 - Spacetime metric

Spacetime metric is a) Non-degenerate ($\det g \neq 0$)

b) Symmetric $(0,2)$ tensor

c) Signature $(-,+,+,+)$

Line element:

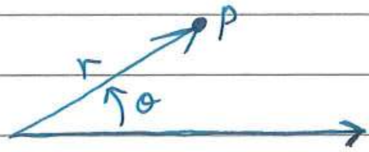
$$ds^2 = g_{ab}(x)\, dx^a dx^b$$

- gives infinitesimal distance between x^a and $x^a + dx^a$.

Example: $\mathbb{R}^2$ + Euclidean metric

$$ds^2 = dx^2 + dy^2$$

$$= dr^2 + r^2 d\theta^2$$



$$\Rightarrow g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

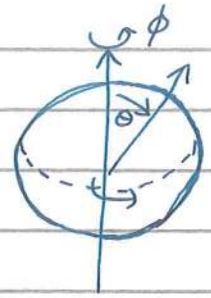
$$g^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}$$

Note: (x,y) are global chart
(r,θ) good for $\mathbb{R}^2 \setminus \{0\}$

Example: S^2 + round metric

$0 \leq \phi < 2\pi$

$0 < \theta < \pi$



(Valid away from $\theta = 0, 2\pi$)

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

$$g^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}$$

As for diccuccion in lectures 2+3

- g_{ab} defines inner product on T_pM

  $g(V, W) = g_{ab} V^a W^b$

- V^a is TL if $g(V, V) < 0$, etc.

- $g^{ab} g_{bc} = \delta^a{}_c$

- g_{ab} defines $T_pM \cong (T_pM)^*$

  $V_a = g_{ab} V^b$, $\omega^a = g^{ab} \omega_b$

6.2 - Levi-Civita covariant derivative

Recall : $\nabla_a \phi = \partial_a \phi$

$$\nabla_a (S^{\cdot\cdot}{}_{\cdot\cdot} + T^{\cdot\cdot}{}_{\cdot\cdot}) = \nabla_a S^{\cdot\cdot}{}_{\cdot\cdot} + \nabla_a T^{\cdot\cdot}{}_{\cdot\cdot}$$

$$\nabla_a (S^{\cdot\cdot}{}_{\cdot\cdot} T^{\cdot\cdot}{}_{\cdot\cdot}) = (\nabla_a S^{\cdot\cdot}{}_{\cdot\cdot}) T^{\cdot\cdot}{}_{\cdot\cdot} + S^{\cdot\cdot}{}_{\cdot\cdot} (\nabla_a T^{\cdot\cdot}{}_{\cdot\cdot})$$

$$\nabla_a V^b = \partial_a V^b + \Gamma^b{}_{ac} V^c$$

(1,1) tensor — not a tensor

The metric g_{ab} determines a unique connection ∇_a

* 1) $\nabla_a g_{bc} = 0$ "metric compatible"

2) $\Gamma^c{}_{ab} = \Gamma^c{}_{ba}$ "torsion free"

Proof : $\nabla_a g_{bc} = \partial_a g_{bc} - \Gamma^d{}_{ab} g_{dc} - \Gamma^d{}_{ac} g_{bd} = 0 \quad (1)$

$\nabla_b g_{ca} = \partial_b g_{ca} - \Gamma^d{}_{bc} g_{da} - \Gamma^d{}_{ba} g_{cd} = 0 \quad (2)$

$\nabla_c g_{ab} = \ldots \quad (3)$

Consider $(2)+(3)-(1)$ and use $g_{ab} = g_{ba}$, $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$

$$\Rightarrow \partial_b g_{ca} + \partial_c g_{ab} - \partial_a g_{bc} = 2\, \Gamma^d{}_{bc} g_{da}$$

$$\Rightarrow \Gamma^a{}_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc} \right)$$

These $\Gamma^a{}_{bc}$ are known as "Christoffel symbols"

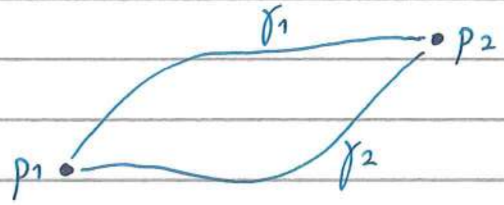
6.3 - Parallel transport

In flat space, $g_{ab} = \eta_{ab}$, can pick $\Gamma^a{}_{bc} = 0$

- Can look at change in a tensor using

$$\partial_a T^b = 0 \Leftrightarrow T^b \text{ constant in } x^a$$

In curved space, $\nabla_a T^b = 0$ is natural generalisation



The way $T^a|_{p_1}$ is transported to p_2 depends on the path taken

- Have to specify the path we are using

For a curve $\gamma = x^a(\lambda)$ with tangent $v^a = \frac{dx^a}{d\lambda}$

$$\nabla_v := \frac{D}{D\lambda} = v^a \nabla_a \equiv \nabla_{\dot{\gamma}}$$

<u>Def</u>: A tensor $S^{a_1 \ldots a_p}{}_{b_1 \ldots b_q}(x)$ is <u>parallel transported</u> along a curve $x^a(\lambda)$ if

$$V^a \nabla_a S^{a_1 \ldots}{}_{b_1 \ldots} = 0$$

i.e. for V^a tangent to $x^a(\lambda)$

$$\nabla_V S^{a_1 \ldots}{}_{b_1 \ldots} = 0$$

This specifies a <u>unique</u> way to move a vector along a curve

i.e. given a vector $u^a \in T_p M$ for $p = \gamma(0)$, there is a unique vector field $U^a(\lambda)$ s.t.

1) $\nabla_{\dot\gamma} U^a = 0$

2) $U^a(0) = u^a$

Consider a curve $\gamma(\lambda)$ with $\dot\gamma = T$ that parallel transports its <u>own</u> tangent vector

$$\begin{aligned} 0 &= \nabla_{\dot\gamma} T^a \\ &= \nabla_T T^a \\ &= T^b \nabla_b T^a \\ &= \frac{dx^b}{d\lambda}\left(\frac{\partial}{\partial x^b} \frac{dx^a}{d\lambda} + \Gamma^a{}_{bc} \frac{dx^c}{d\lambda} \right) \end{aligned}$$

$$= \frac{d^2x^a}{d\lambda^2} + \Gamma^a{}_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda}$$

$$= \ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c$$

Such a curve $x^a(\lambda)$ is called a geodesic.

Note: we could have taken

$$T^b \nabla_b T^a = \alpha T^a$$

as our defintion.

We can always reparametrise our curve $\lambda \mapsto \lambda'(\lambda)$ such that

$$g_{ab}T^aT^b = T_aT^a = \text{const. as fn. } \lambda$$

Thus

$$\frac{d}{d\lambda}(T^aT_a) = 0$$

$$= \frac{dx^b}{d\lambda}\partial_b(T^aT_a)$$

$\partial \equiv \nabla$ on scalars

$$= T^b \nabla_b (T^aT_a)$$

$$= 2\underbrace{(T^b D_b T^a)}_{\alpha T^a} T_a$$

$$= 2\alpha T^a T_a$$

$$\Rightarrow \quad \alpha = 0$$

A parametrisation in which $T_a T^a = \text{const}$ is called affine.

- The geodesic equation is

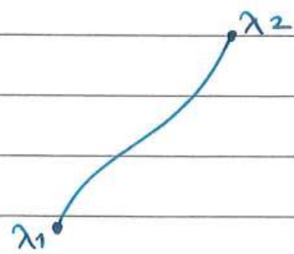
$$T^a D_a T^b = 0$$

- Affine parameters related by

$$\lambda = a\lambda' + b$$

Timelike geodesics minimise proper time τ.

$$\Delta\tau = \int_{\lambda_1}^{\lambda_2} d\lambda \sqrt{-g_{ab}(x)\,\dot{x}^a \dot{x}^b}$$



$$\dot{x}^a = \frac{dx^a}{d\lambda}$$

<u>Proof</u>: for a timelike geodesic, $\sqrt{-g(\dot{x},\dot{x})} > 0$ along curve so <u>equivalent</u> to minimising

$$S = \int_{\lambda_1}^{\lambda_2} \mathcal{L} \,, \quad \mathcal{L} = g_{ab}(x)\,\dot{x}^a \dot{x}^b$$

<u>NB</u>: $\Delta\tau$ functional invariant under any reparametrisation of λ.

S invariant under $\lambda \mapsto a\lambda + b$ "affine"

S minimised for $\mathcal{L}$ satisfying <u>Euler-Lagrange</u> equations

EL: $$\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^a} - \frac{\partial \mathcal{L}}{\partial x^a} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x^a} = \partial_a g_{bc}\, \dot{x}^b \dot{x}^c$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^a} = 2\, g_{ab}\, \dot{x}^b$$

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\right) = 2\left(\frac{d}{d\lambda} g_{ab}\right)\dot{x}^b + 2\, g_{ab}\, \ddot{x}^b$$

$$= 2\,\dot{x}^c \partial_c g_{ab}\, \dot{x}^b + 2\, g_{ab}\, \ddot{x}^b$$

$$\Rightarrow 0 = g_{ab}\ddot{x}^b + \partial_c g_{ab}\dot{x}^c\dot{x}^b - \frac{1}{2}\partial_a g_{bc}\dot{x}^b\dot{x}^c$$

$$= g_{ab}\ddot{x}^b + \frac{1}{2}\left(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc}\right)\dot{x}^b\dot{x}^c$$

$$\Rightarrow 0 = \ddot{x}^a + \frac{1}{2}g^{ad}\left(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}\right)\dot{x}^b\dot{x}^c$$

$$\Rightarrow \quad \ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0$$

Comments : 1) Levi-Civita connection appears naturally

2) In practice, comparing geodesic equation with $\delta S = 0$ equations is easiest way to compute $\Gamma^a{}_{bc}$.

Example : Flat $\mathbb{R}^2$ in polar coordinates (PS2)

Example : S^2 with round metric

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

$$\mathcal{L} = \dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \partial_\theta \mathcal{L} = 2\sin\theta\cos\theta\,\dot{\phi}^2 \qquad \partial_\phi \mathcal{L} = 0$$

$$\partial_{\dot{\theta}}\mathcal{L} = 2\dot{\theta} \qquad \partial_{\dot{\phi}}\mathcal{L} = 2\sin^2\theta\,\dot{\phi}^2$$

θ : $\quad \ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0$

ϕ : $\quad \dfrac{d}{d\lambda}\left(\sin^2\theta\,\dot{\phi}\right) = 0 \qquad$ "conservation of ang. momentum"

$$\Rightarrow \quad \ddot{\phi} + 2\frac{\cos\theta}{\sin\theta}\dot{\phi}\dot{\theta} = 0$$

Compare with $\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0$

$$\Rightarrow \quad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \frac{\cos\theta}{\sin\theta}$$

Comment : $\mathcal{L}$ did not depend on ϕ so there was a conserved quantity along the geodesic

What do the geodesics look like?

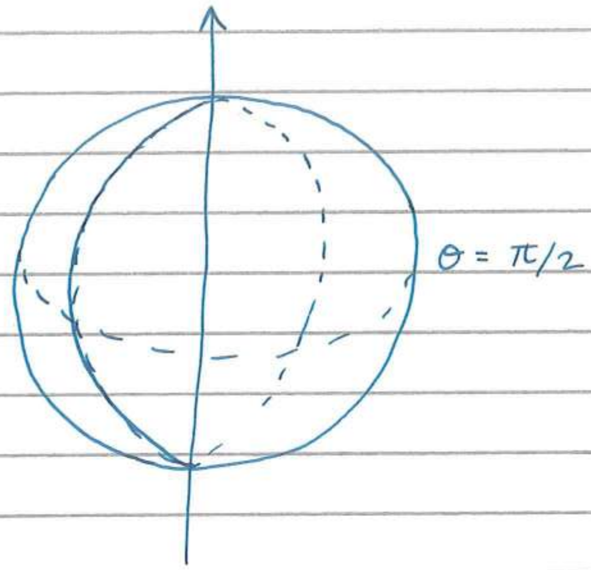
- Choose $\dot{\phi}(0) = 0$ (no ϕ motion)

$\Rightarrow \quad \ddot{\phi} = 0$ from ϕ equation

$\Rightarrow \quad \phi(\lambda) = $ constant (as $\dot{\phi}(0) = \dot{\phi}(\lambda) = 0$)

$\Rightarrow \quad \ddot{\theta} = 0$ from θ equation

So $\quad \dot{\theta} =$ constant $\qquad$ "Great circles"



7.1 - Review

Spacetime : 1) Smooth manifold M

2) Metric tensor $g_{ab}(x)$

Unique covariant derivative ∇_a : 1) $\nabla_a g_{bc} = 0$

2) $\Gamma^c{}_{ab} = \Gamma^c{}_{ba}$

Geodesics : 1) Curve $x^a(\lambda)$ with $T^a = \dot{x}^a(\lambda)$

$$T^a \nabla_a T^b = 0$$

2) Minimise proper time $\Delta\tau$

7.2 - Local inertial frames

Can always find coordinates around a point P such that

1) $x^a(p) = 0$

2) $\Gamma^a{}_{bc}(x) = 0 + O(x)$

3) $g_{ab}(x) = \eta_{ab} + O(x^2)$

Example : S^2 with round metric

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

let $\tilde{\theta} = \theta - \pi/2$, $\tilde{\phi} = \phi - c$

So $(\tilde{\theta}, \tilde{\phi}) = (0,0)$ at $\phi = c$ on equator

$$\Rightarrow ds^2 = d\tilde{\theta}^2 + \cos^2\tilde{\theta} \, d\tilde{\phi}^2$$

$$= d\tilde{\theta}^2 + \left(1 - \frac{\tilde{\theta}^2}{2}\right)^2 d\tilde{\phi}^2$$

$$= d\tilde{\theta}^2 + d\tilde{\phi}^2 + \mathcal{O}(\tilde{\theta}^2)$$

Metric looks flat to $\mathcal{O}(x^2)$

In local inertial frame, geodesic equation is

$$\frac{d^2x^a}{d\lambda^2} = 0 + \mathcal{O}(x^3)$$

so free particles move in straight lines (as in SR).

These local coordinates are those used by a freely falling observer.

In a local frame, $\nabla_a = \partial_a$.

Suggests that we can promote equations from SR to GR using

$$\partial_a \to D_a \qquad \text{"minimal coupling"}$$

which then reduce to SR laws in a local inertial frame.

Example: $D_a F^{ab} = 4\pi J^b$

Electromagnetism

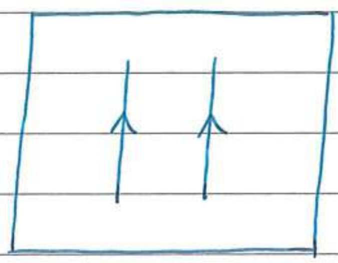
$D_{[a} F_{bc]} = 0$

$D_a T^{ab} = 0$ Stress tensor

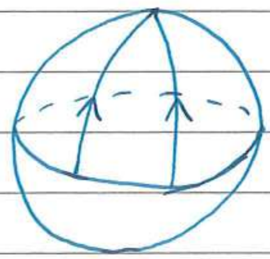
7.3 - Curvature

Deviations from special relativity at $\mathcal{O}(x^2)$ in g_{ab}

Characterised by focussing (or parting) of geodesics



$\mathbb{R}^2$



S^2

Suggests that we can promote equations from SR to GR using

$$\partial_a \to D_a \qquad \text{"minimal coupling"}$$

which then reduce to SR laws in a local inertial frame.

Example: $D_a F^{ab} = 4\pi J^b$

Electromagnetism

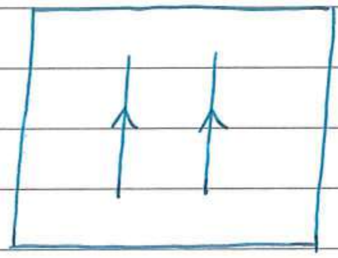
$$D_{[a} F_{bc]} = 0$$

$$D_a T^{ab} = 0 \qquad \text{Stress tensor}$$

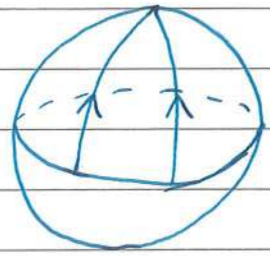
7.3 - Curvature

Deviations from special relativity at $\mathcal{O}(x^2)$ in g_{ab}

Characterised by focussing (or parting) of geodesics



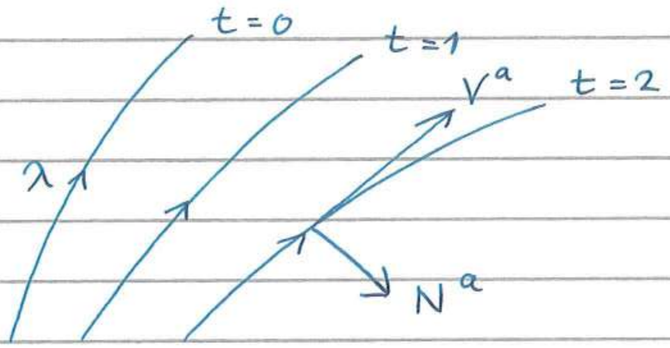
$\mathbb{R}^2$



S^2

Consider a 1-parameter family of timelike geodesics

$$x^a_t(\lambda) = x^a(\lambda, t)$$



$$V^a = \frac{dx^a}{d\lambda}, \qquad N^a = \frac{dx^a}{dt}$$

"Geodesics" $\Rightarrow$ $V^a \nabla_a V^b = 0$ for all t.

We can change the affine parameter

$$\lambda \mapsto a(t)\lambda + b(t)$$

$$\Rightarrow \quad V^a \mapsto \frac{1}{a(t)} V^a$$

$$N^a \mapsto N^a + (a'(t)\lambda + b'(t)) V^a$$

Fix this using: 1) Norm of V^a preserved with t

2) Set $g(N, V) = 0$ (ignore deviation along geodesic)

1) $\frac{d}{d\lambda} g(V,V) = V^a \partial_a g(V,V)$ scalar

$= V^a \nabla_a g(V,V)$ "metric" ∇_a

$= 2 V^a g_{bc} \nabla_a V^b V^c$

$= 2 V_b \underbrace{V^a \nabla_a V^b}_{0}$

$= 0$

$\Rightarrow g(V,V) = -f(t)^2$ for some $f(t)$

Choose $a(t) = f(t)$ to fix

$g(V,V) = -1$

so that λ is proper time

2) $\frac{d}{d\lambda} g(V,N) = V^c \nabla_c (g_{ab} V^a N^b)$ $\nabla g = 0$, $V \nabla V = 0$

$= g_{ab} V^a V^c \nabla_c N^b$

but $V^c \nabla_c N^b - N^c \nabla_c N^b = V^c \partial_c N^b - N^c \partial_c V^b$

need this $= \frac{\partial}{\partial\lambda}\frac{dx^a}{dt} - \frac{\partial}{\partial t}\frac{\partial x^a}{\partial\lambda}$

$= 0$

So $$\frac{d}{d\lambda} g(V,N) = g_{ab} V^a N^c \nabla_c V^b$$

$$= \frac{1}{2} N^c \nabla_c \underbrace{\left(g_{ab} V^a V^b \right)}_{= -1 \ \forall \lambda, t}$$

$$= 0$$

So $g(V,N)$ constant with λ.

Use remaining $\lambda \mapsto \lambda + b(t)$

$$N^a \mapsto N^a + b'(t) V^a$$

to fix $g(V,N) = 0$.

<u>Relative acceleration</u> of nearby geodesics

$$\frac{D^2}{D\lambda^2} N^a = \nabla_V \nabla_V N^a$$

$$= V^b \nabla_b (V^c \nabla_c N^a)$$

$$\downarrow \ [V,N]^a = 0$$

$$= V^b \nabla_b (N^c \nabla_c V^a)$$

$$= V^b (\nabla_b N^c)(\nabla_c V^a) + V^b N^c \nabla_b \nabla_c V^a$$

$$= V^b (\nabla_b N^c)(\nabla_c V^a)$$

$$+ V^b N^c (\nabla_b \nabla_c - \nabla_c \nabla_b) V^a$$

$$+ V^b N^c \nabla_c \nabla_b V^a$$

$$[N,V]=0$$

$$= N^b \nabla_b V^c \nabla_c V^a + V^b N^c \nabla_c \nabla_b V^a$$

$$+ V^b N^c [\nabla_b, \nabla_c] V^a$$

= 0 as geodesic.

$$= N^b \nabla_b (V^c \nabla_c V^a)$$

$$+ V^b N^c [\nabla_b, \nabla_c] V^a$$

$$\equiv + R_{bc}{}^a{}_d V^b N^c V^d$$

This defines the Riemann curvature tensor

$$+ R_{bc}{}^a{}_d V^d = [\nabla_b, \nabla_c] V^a$$

Clearly C^∞ linear in bc indices

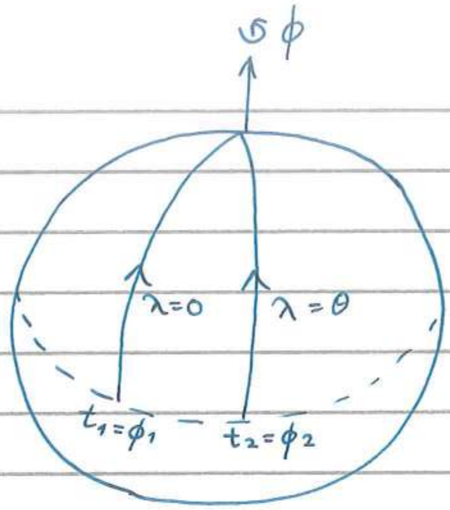
- Can check it is linear under

$$V^a \to f V^a \qquad (\text{c.f. } [\nabla_a, \nabla_b] f = 0 \text{ for torsion free})$$

so transforms as a tensor.

Example: Round S^2

Identify $t = \phi$, $\lambda = \theta$



$$V^a = (V^\theta, V^\phi) = (1, 0)$$

$$N^a = (N^\theta, N^\phi) = (0, 1)$$

Recall : $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$

$$\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \frac{\cos\theta}{\sin\theta}$$

$$\frac{DN^a}{D\lambda} = \frac{dx^b}{d\lambda} \nabla_b N^a$$

$$= V^b \nabla_b N^a$$

$$= \underbrace{\partial_\theta N^a}_{0} + \Gamma^a{}_{\theta\phi} N^\phi$$

$$\Rightarrow \frac{DN^\phi}{D\lambda} = \Gamma^\phi{}_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$$

Now $\quad \dfrac{D^2 N^\phi}{D\lambda^2} = \partial_\theta\left(\dfrac{DN^\phi}{D\lambda}\right) + \Gamma^\phi{}_{\theta\phi}\dfrac{DN^\phi}{D\lambda}$

$$= \partial_\theta\left(\frac{\cos\theta}{\sin\theta}\right) + \left(\frac{\cos\theta}{\sin\theta}\right)^2$$

$$= -1$$

$$\equiv + R_{bc}{}^{\phi}{}_{d}\, V^b N^c V^d$$

$$= + R_{\theta\phi}{}^{\phi}{}_{\theta}$$

So $\quad R_{\theta\phi}{}^{\phi}{}_{\theta} = -1$

<u>Comment</u>: In 2d, Riemann tensor has only 1 independent component (see PS3)

8.1 – Riemann tensor

Recall $[\nabla_a, \nabla_b] V^d = R_{ab}{}^d{}_c V^c$

Extends to other tensors by induction

1) $[\nabla_a, \nabla_b]\phi = [\partial_a, \partial_b]\phi - (\Gamma^c{}_{ab} - \Gamma^c{}_{ba})\partial_c\phi$

$$= 0$$

2) Apply to $\phi = V^a \omega_a$ to read off action

$$[\nabla_a, \nabla_b] T^{c_1 \dots}{}_{d_1 \dots} = R_{ab}{}^{c_1}{}_e T^{e c_2 \dots}{}_{d_1 \dots} + \dots$$

$$- R_{ab}{}^e{}_{d_1} T^{c_1 \dots}{}_{e d_2 \dots} + \dots$$

(Think of $R_{ab}{}^c{}_d$ as $(R_{ab})^c{}_d$, then acts with + on vector index and − on covectors)

Expression for $R_{ab}{}^c{}_d$ in terms of g_{ab}

$$[\nabla_a, \nabla_b] V^c = \partial_a \nabla_b V^c - \Gamma^d{}_{ab} \nabla_d V^c + \Gamma^c{}_{ad} \nabla_b V^d$$

$$- (a \leftrightarrow b)$$

$$= \partial_a(\partial_b V^c + \Gamma^c{}_{bd} V^d) - \Gamma^d{}_{ab}(\partial_d V^c + \Gamma^c{}_{de} V^e)$$

$$+ \Gamma^c{}_{ad}(\partial_b V^d + \Gamma^d{}_{be} V^e) - (a \leftrightarrow b)$$

Simplify using $[\partial_a, \partial_b] = 0$ and $\Gamma^a{}_{[bc]} = 0$

- All derivatives of V drop out

$$= \left(\partial_a \Gamma^c{}_{be} + \Gamma^c{}_{ad} \Gamma^d{}_{be} - a \leftrightarrow b\right) V^e$$

$$\equiv R_{ab}{}^c{}_e V^e$$

Comments : 1) Explicitly linear in $V^e \Rightarrow$ tensorial (verifying using transformation law is painful).

2) Depends on Γ and $\partial \Gamma$

3) Can pick $\Gamma|_p = 0$ but $\partial \Gamma|_p \neq 0$, so R does not vanish in local inertial frame in general.

4) $R = 0$ in cartesian for Minkowski, so also zero in any coordinates - converse ~~also~~ also holds!

8.2 - Algebraic identities

1) $R_{ab}{}^c{}_d = - R_{ba}{}^c{}_d$ (from definition)

2) $R_{abcd} = - R_{abdc}$ (∇_a metric compatible)

Since $\nabla_a g_{bc} = 0$

$$0 = [\nabla_a, \nabla_b] g_{cd}$$

$$= R_{abcd} + R_{abdc}$$

3) $R_{[abc]d} = 0$ "First Bianchi identity"

($\text{Torsion free } \Gamma^a{}_{[bc]} = 0$)

Compute $\nabla_{[a} \nabla_b \nabla_{c]} \phi = 0$ (uses torsion free)

But equal to $R_{[ab}{}^d{}_{c]} \nabla_d \phi$

As this vanishes for all ϕ, $R_{[ab}{}^d{}_{c]} = 0$

$\Rightarrow R_{[abc]d} = 0$

1) + 2) + 3) $\Rightarrow$ $R_{abcd} = R_{cdab}$

8.3 - Bianchi identity

$\nabla_{[a} R_{bc]}{}^d{}_e = 0$ (like $\nabla_{[a} F_{bc]} = 0$)

(needs torsion free)

1) $(\nabla_a \nabla_b - \nabla_b \nabla_a) \nabla_c \omega_d$

$= - R_{ab}{}^e{}_c \nabla_e \omega_d - R_{ab}{}^e{}_d \nabla_c \omega_e$

2) $\nabla_c (\nabla_a \nabla_b - \nabla_a \nabla_b) \omega_d$

$= \nabla_c (-R_{ab}{}^e{}_d \omega_e)$

$= - \nabla_c R_{ab}{}^e{}_d \omega_e - R_{ab}{}^e{}_d \nabla_c \omega_e$

Antisymmetrise 1) and 2) over a, b, c

- LHS are equal so

$$-\underbrace{R_{[ab}{}^{e}{}_{c]}}_{\text{vanishes due to 1st Bianchi}} \nabla_e \omega_d - R_{[ab|}{}^{e}{}_{d} \nabla_{|c]} \omega_e$$

cancel

$$= -\underbrace{\nabla_{[c} R_{ab]}{}^{e}{}_{d}}_{\text{must vanish for all } \omega_e} \omega_e - R_{[ab|}{}^{e}{}_{d} \nabla_{|c]} \omega_e$$

8.4 - Einstein equation

So far we have described the effects of a gravitational field on a test particle

- Like $\ddot{\underline{x}} = -\underline{\nabla}\Phi$

Gravity is non-linear, the gravitational field should satisfy a generalisation of the Poisson equation, $\nabla^2\Phi \sim 4\pi G\rho$

That is, for a given distribution of matter and energy, what is the resulting metric?

Should be of the form

$$\text{Tensor}(g_{ab}) = \text{Tensor}(\text{matter, energy}\ldots)$$

Stress tensor T^{ab} captures source contribution

$$\Rightarrow \quad G_{ab} = \lambda T_{ab}$$

where – G_{ab} linear in $R_{ab}{}^{c}{}_{d}$ (2^{nd} order equations of motion for the metric)

– $G_{ab} = G_{ba}$ (as T symmetric)

– $\nabla^a G_{ab} = 0$ (as $\nabla^a T_{ab} = 0$)

($G_{ab} \propto g_{ab}$ doesn't work as $\mathrm{tr}\, g = 2$ but $\mathrm{tr}\, T_{ab}$ can vanish)

There is (almost) a unique answer to this question!

Define: $R_{ab} = R_{acb}{}^{c} = R_{acbd}\, g^{cd}$

"Ricci tensor" ($R_{ab} = R_{ba}$)

$$R = g^{ab} R_{ab}$$

"Ricci scalar"

Now start from Bianchi identity and contract

$$\nabla_{[a} R_{bc]de} = 0$$

$$\equiv \quad \nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{abde} = 0$$

$\times \; g^{bd} g^{ce}$ and use $\nabla g = 0$

$$= \nabla_a R_{bc}{}^{bc} + \nabla_b R_{ca}{}^{bc} + \nabla_c R_{ab}{}^{bc}$$

$$= \nabla_a R - \nabla_b R_a{}^b - \nabla_c R_a{}^c$$

$$= 0$$

$$\Rightarrow \quad \nabla^a (g_{ab} R - 2 R_{ab}) = 0$$

i.e. $$\nabla^a \underbrace{\left(R_{ab} - \tfrac{1}{2} g_{ab} R \right)}_{G_{ab}} = 0$$

G_{ab} known as <u>Einstein tensor</u>.

$$R_{ab} - \tfrac{1}{2} g_{ab} = \lambda T_{ab}$$

<u>"Einstein equation"</u>

Fix λ using Newtonian limit.

<u>Example</u> : S^2 with round metric

Recall $R_{\theta\phi}{}^{\phi}{}_{\theta} = -1$

$$g^{\theta\theta} = 1 \qquad g_{\theta\theta} = 1$$

$$g^{\phi\phi} = \frac{1}{\sin^2\theta} \qquad g_{\phi\phi} = \sin^2\theta$$

$$R_{\theta\theta} = R_{\theta c \theta}{}^{c}$$

$$= R_{\theta\phi\theta}{}^{\phi}$$

$$= +1$$

$$R_{\phi\phi} = R_{\phi\theta\phi}{}^{\theta}$$

$$= R_{\phi\theta}{}^{\phi}{}_{\theta}\, g_{\phi\phi}\, g^{\theta\theta}$$

$$= (+1)\sin^2\theta$$

i.e. $R_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$

Note $R_{ab} = g_{ab}$! "<u>Einstein metric</u>"

Scalar $R = g^{ab} R_{ab} = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi}$

$$= 2$$

Einstein tensor: $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$

$= 0$ identically.

True for any metric in two dimensions.

Comments: 1) $R \sim \partial\Gamma + \Gamma^2$

$\sim \partial^2 g + \partial g \, \partial g$

R is 2^{nd} order in derivatives, as is G_{ab}.

Thus the equation for g_{ab} is 2^{nd} and g_{ab} is a dynamical field.

2) One can derive Einstein equations from a 4d action

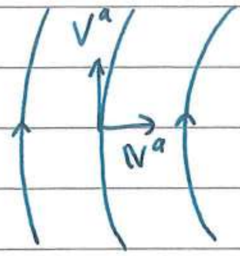
$$S = \int d^4x \sqrt{-g} \; R$$

$\delta S = 0$ reproduces $G_{ab} = 0$

Can add "matter fields to generate T_{ab}.

9.1 - Review

Geodesic deviation : $\frac{D^2}{D\lambda^2} N^a = R_{bc}{}^a{}_d V^b N^c V^d$



Einstein equations : $R_{ab} - \frac{1}{2} g_{ab} = \lambda T_{ab}$

Plan : Determine λ by going to Newtonian limit and comparing to Newton's 2nd law.

9.2 - Newtonian limit : curvature

Assume existence of appnx. global frame with :

1) $g_{ab} = \eta_{ab} + \varepsilon h_{ab} + \mathcal{O}(\varepsilon^2)$

2) $\partial_t g_{ab} = 0 + \mathcal{O}(\varepsilon)$

Gravity is : 1) Weak.

2) Slowly varying.

View h_{ab} as perturbation around flat space.

- To $\mathcal{O}(\varepsilon)$, raise and lower with η_{ab}, η^{ab}.

Now want to compute curvature

1) Inverse metric: $g_{ab} = \eta_{ab} + \varepsilon h_{ab}$

$$\Rightarrow g^{ab} = \eta^{ab} - \varepsilon h^{ab}$$

so that $g_{ab} g^{bc} = \delta_a^{\ c} + \mathcal{O}(\varepsilon^2)$

2) Christoffel symbols

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})$$

$$= \frac{1}{2} \varepsilon \eta^{ad} (\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc}) + \mathcal{O}(\varepsilon^2)$$

3) Riemann tensor: since $\Gamma \sim \mathcal{O}(\varepsilon)$, only $\partial \Gamma$ is $\mathcal{O}(\varepsilon)$ (ignore Γ^2)

$$R_{ab}{}^c{}_d = \partial_a \Gamma^c{}_{bd} - \partial_b \Gamma^c{}_{ad} + \mathcal{O}(\varepsilon^2)$$

$$= \frac{1}{2} \varepsilon \Big[\eta^{ce} (\partial_a \partial_b h_{de} + \partial_a \partial_d h_{be} - \partial_a \partial_e h_{bd})$$

$$\updownarrow 0$$

$$- \eta^{ce} (\partial_b \partial_a h_{de} + \partial_b \partial_d h_{ae} - \partial_b \partial_e h_{ad}) \Big]$$

$$R_{abcd} = \frac{\varepsilon}{2}\Big(\partial_a \partial_d h_{bc} - \partial_a \partial_c h_{bd}$$
$$+ \partial_b \partial_c h_{ad} - \partial_b \partial_d h_{ac} \Big) + \mathcal{O}(\varepsilon^2)$$

Comment: symmetries are manifest.

4) Ricci tensor:

$$R_{ac} = g^{bd} R_{abcd}$$
$$= \eta^{bd} R_{abcd} + \mathcal{O}(\varepsilon^2)$$
$$= \frac{\varepsilon}{2}\Big(\partial_a \partial_b h^b{}_c - \partial_a \partial_c h$$
$$+ \partial_b \partial_c h_a{}^b - \partial_b \partial^b h_{ac} \Big)$$

$$= \frac{\varepsilon}{2}\Big(2\, \partial^b \partial_{(a} h_{c)b} - \partial_b \partial^b h_{ac} - \partial_a \partial_c h \Big)$$

where $h := \eta^{ab} h_{ab} = h^a{}_a = h_a{}^a$

5) Einstein tensor: introduce "trace-free" perturbation (or trace reversed)

$$\bar{h}_{ab} = h_{ab} - \tfrac{1}{2} \eta_{ab} h \qquad (\operatorname{tr} \eta_{ab} = 2)$$

Then: $\bar{h} = h - 2h = -h \Rightarrow h_{ab} = \bar{h}_{ab} - \tfrac{1}{2} \eta_{ab} \bar{h}$

$$G_{ac} := R_{ac} - \frac{1}{2} g_{ac} R$$

$$= \frac{\varepsilon}{2}\left(2\, \partial^b \partial_{(a} \bar{h}_{c)b} - \partial_b \partial^b \bar{h}_{ac} - \eta_{ac} \partial^b \partial^d \bar{h}_{bd} \right)$$

6) Gauge invariance

$g_{ab} = \eta_{ab} + \varepsilon h_{ab} + O(\varepsilon^2)$ does not fully specify the coordinate system.

- There can be other coordinates in which $g = \eta + h + \ldots$ but h will be different.
- Decomposition of metric into background + perturbation is not unique.
- Different h's can give same curvature, i.e. same physical spacetime to $O(\varepsilon^2)$

Gauge transformation:

$$x^a \mapsto x^a + \xi^a(x)$$

$$h_{ab} \mapsto h_{ab} - \partial_a \xi_b - \partial_b \xi_a$$

Comment: can check $\delta R_{abcd} = 0$ to $O(\xi^2)$
so h'_{ab} describes same geometry.

Want to match to Newton

- Pick "harmonic" gauge (nice coordinates)
- Also known as Einstein / Hilbert / de Donder gauge.

$$\partial^a \bar{h}_{ab} = 0$$

- Weak field limit of

$$g^{ab} \Gamma^c{}_{ab} = 0 \iff \nabla_a \partial^a x^b = \Box x^b = 0$$

not a vector

(Cartesian coordinates in SR solve this too!)

$$\Rightarrow G_{ac} = -\frac{\varepsilon}{2} \partial_b \partial^b \bar{h}_{ac} + O(\varepsilon^2)$$

$$= -\frac{\varepsilon}{2} \underline{\nabla}^2 \bar{h}_{ac} + O(\varepsilon^2)$$

9.3 - Newtonian limit: Einstein equation

Assume pressureless matter at rest in the approx. inertial frame

$$T_{ab} = \rho u_a u_b, \qquad u_a = (1, 0, 0, 0)$$

So $\quad G_{ab} = \lambda T_{ab} \quad$ is

$$-\frac{1}{2} \varepsilon \underline{\nabla}^2 \bar{h}_{ab} = \lambda T_{ab}$$

$$\begin{cases} -\frac{1}{2} \varepsilon \underline{\nabla}^2 \bar{h}_{00} = \lambda \rho \\ \underline{\nabla}^2 \bar{h}_{ab} = 0 \qquad (ab) \neq (00) \end{cases}$$

2^{nd} equation: if $\bar{h}_{ab} \to 0$ at infinity, need

$$\bar{h}_{ab} = 0 \qquad (ab) \neq (00)$$

$$\Rightarrow \bar{h} = \eta^{ab} \bar{h}_{ab} = -\bar{h}_{00}$$

$$\Rightarrow h_{00} = \bar{h}_{00} - \frac{1}{2} \eta_{00} \bar{h}$$

$$= \frac{1}{2} \bar{h}_{00}$$

So 1^{st} eq^{n} is: $\quad \underline{\nabla}^2 h_{00} = -\frac{\lambda}{\varepsilon} \rho$

Compare : $\underline{\nabla}^2 \phi = 4\pi \rho G$

- how is ϕ related to $\bar{h}_{00}$?

9.4 - Newtonian limit : geodesic equation.

$$\ddot{x}^a + \Gamma^a{}_{bc}\, \dot{x}^b \dot{x}^c = 0$$

$\dot{x}^a = \dfrac{dx^a}{d\tau}$ "proper time reduces to coordinate time in Newtonian limit"

Assume non-relativistic motion :

$$\dot{x}^a = (1, 0, 0, 0) + \mathcal{O}(\varepsilon)$$

$\tau \simeq$ coordinate time x^0 of approx. inertial frame.

$$\Rightarrow \frac{d^2 x^a}{dt^2} = -\Gamma^a{}_{00}$$

$$\Rightarrow \frac{d^2 x^i}{dt^2} = -\Gamma^i{}_{00} = +\frac{\varepsilon}{2}\partial_i h_{00}$$

Compare : $\ddot{x}^i = -\partial_i \phi \quad \Rightarrow \quad \phi = -\frac{\varepsilon}{2} h_{00} = -\frac{\varepsilon}{4}\bar{h}_{00}$

9.5 - Value of λ

Plug back into Einstein equation

$$\Rightarrow \quad \underline{\nabla}^2 \phi = \frac{\lambda}{2} \rho = 4\pi G \rho$$

$$\Rightarrow \quad \lambda = 8\pi G$$

$$\Rightarrow \quad R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G \, T_{ab}$$

10 - Schwarzschild Solution

10.1 - Vacuum equations

Absence of mass / energy : $T_{ab} = 0$

$$\Rightarrow \quad R_{ab} - \frac{1}{2} g_{ab} R = 0 \qquad \downarrow g^{ab}$$

$$\Rightarrow \quad R - 2R = 0$$

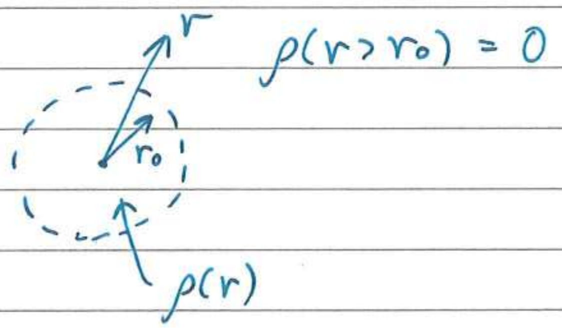
$$\Rightarrow \quad R_{ab} = 0$$ "vacuum equations"

10.2 - Schwarzschild solution.

Gravitational field outside spherical distribution of matter.

Birkhoff's theorem: ~~any~~ there is a unique static, spherically symmetric solution of $R_{ab} = 0$

(spherically $\Rightarrow$ static + asymp. flat $\Rightarrow$ Schwarzschild)



Metric outside :

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2$$

$$+ r^2 \underbrace{\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)}_{\substack{\text{slice at fixed } (t,r) \\ \text{has } S^2 \text{ geometry}}}$$

r : radial coordinate

t : time coordinate

$R_s = 2GM$ is Schwarzschild radius

Restoring units : $R_s = \dfrac{2GM}{c^2}$

- Sun : $R_s \simeq 1$ km
- Earth : $R_s \simeq 1$ cm

Astrophysics cares about $r > R_s$ region.

Newtonian limit: 1) Weak field for $r \gg R_s$.

2) $\frac{dt}{dt} \gg \frac{dr}{d\tau}$ non-relativistic.

$$g_{tt} = -\left(1 - \frac{2GM}{r}\right)$$

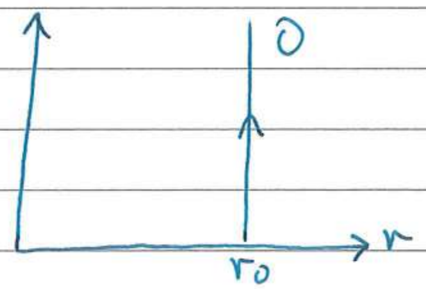
$$\Rightarrow h_{00} = \frac{2GM}{r}$$

Compare with $\phi = -\frac{1}{2} h_{00} = -\frac{GM}{r}$

from last lecture

10.3 - Stationary observers

Stationary: $r = r_0 = $ const.



Question: how is proper time of O related to t?

t is "coordinate time".

Since $dr = 0$ for $\mathcal{O}$ (and $d\theta = d\phi = 0$)

$$ds^2 = -d\tau^2 = -\left(1 - \frac{2GM}{r_0}\right) dt^2$$

$$\Rightarrow \quad \Delta\tau = \left(1 - \frac{2GM}{r_0}\right)^{1/2} \Delta t$$

As $r_0 \gg R_s$, $\Delta\tau \to \Delta t$

Coordinate time t is the proper time of a stationary observer at spatial infinity $r \to \infty$.

Alternatively: $\mathcal{O}$ is stationary

$$u^a = \frac{dx^a}{d\tau} = \left(\frac{dt}{d\tau}, 0, 0, 0\right)$$

$$u_a u^a = -1$$

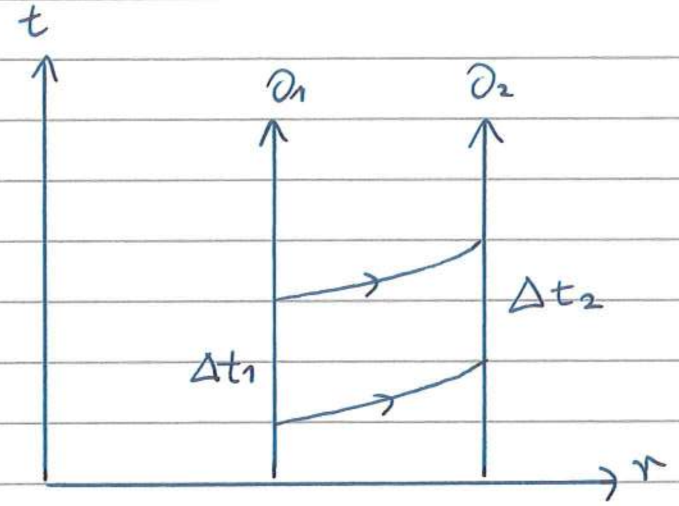
$$= g_{00}(u^0)^2$$

$$= -\left(1 - \frac{2GM}{r_0}\right)\left(\frac{dt}{d\tau}\right)^2$$

$$\Rightarrow \quad \frac{dt}{d\tau} = \frac{1}{\left(1 - \frac{2GM}{r_0}\right)^{1/2}}$$

10.4 - Gravitational Redshift

Consider 2 stationary observers $\mathcal{O}_1$ and $\mathcal{O}_2$



- $\mathcal{O}_1$ emits signals separated by Δt_1.
- $\mathcal{O}_2$ receives signals separated by Δt_2

Spacetime is static (metric does not depend on t)

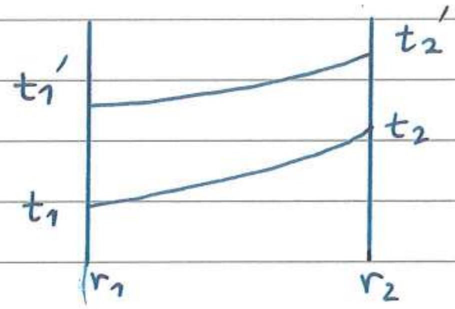
$$\Rightarrow \quad \Delta t_1 = \Delta t_2 = \Delta t \text{ "coordinate time agrees"}$$

Explicitly: light travels on null geodesic, a curve that obeys

$$ds^2 = 0$$

$$\Rightarrow \left(1 - \frac{2MG}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 = 0$$

$$\Rightarrow \quad \frac{dt}{dr} = + \frac{1}{\left(1 - \frac{2GM}{r}\right)^{1/2}}$$ (+ as t increases with r, outgoing)



$$\int_{t_1}^{t_2} dt = \int_{r_1}^{r_2} \frac{1}{\left(1 - \frac{2GM}{r}\right)^{1/2}} = \int_{t_1'}^{t_2'} dt$$

$$\Rightarrow \quad t_2 - t_1 = t_2' - t_1'$$

$$\Rightarrow \quad \Delta t_1 = \Delta t_2$$

How much proper time passes for $\mathcal{O}_1$ and $\mathcal{O}_2$?

Stationary $\Rightarrow$ $dr = d\theta = d\phi = 0$

$$\Rightarrow \quad -d\tau^2 = -\left(1 - \frac{2GM}{r}\right) dt^2$$

$$\Rightarrow \quad d\tau = \left(1 - \frac{2GM}{r}\right)^{1/2} dt$$

$$\Rightarrow \quad \Delta \tau_i = \left(1 - \frac{2GM}{r_i}\right)^{1/2} \Delta t$$

So $$\frac{\Delta\tau_2}{\Delta\tau_1} = \left(\frac{1 - \frac{2GM}{r_2}}{1 - \frac{2GM}{r_1}}\right)^{1/2}$$

As $r_2 > r_1$, $\Delta\tau_2 > \Delta\tau_1$

Less time passes for $\mathcal{O}_1$ than $\mathcal{O}_2$

$\mathcal{O}_1$'s clock "runs slow"

Clocks run slow in gravitational field / curved spacetime.

"Gravitational Time Dilation".

Example: Near surface of Earth

$$r_1 = R_E$$

$$r_2 = R_E + h \quad , \quad h \ll R_E$$

$$\frac{\Delta\tau_2}{\Delta\tau_1} \simeq 1 - \frac{GM}{R_E+h} + \frac{GM}{R_E} \quad (\text{expand } 1/2)$$

$$\simeq 1 + \frac{GM}{R_E^2} h$$

$$= 1 + gh \quad \text{"shift in photon } \omega\text{"}$$

Consider two different limits

1) $r_2 \to \infty$, $\mathcal{O}_2$ far away from source

$$\frac{\Delta\tau_2}{\Delta\tau_1} \to \frac{1}{\left(1 - \frac{2GM}{r_1}\right)^{1/2}} \qquad \circledast$$

$$\Delta\tau_2 = \Delta t$$

Can be used to operationally define r.

Stationary observer $\mathcal{O}_2$ at ∞ sends signals to $\mathcal{O}_1$ separated by $\Delta\tau_2$.

$\mathcal{O}_1$ sends message back with how long between signals ($\Delta\tau_1$)

$\mathcal{O}_2$ then defines $\mathcal{O}_1$ to be at r_1 given by $\circledast$

2) $r_1 \to 2GM$

$$\frac{\Delta\tau_2}{\Delta\tau_1} \to \infty \,!$$

$\mathcal{O}_2$ observes $\mathcal{O}_1$ to slow down and eventually "freeze" as they pass the Schwarzschild radius.

NB: For stars, etc, solution only valid for $r > R_E$ and $R_E \gg 2GM_E$, so do not see this!

radius of body

11.1 – Geodesics for Schwarzschild

Lagrangian for affinely parametrised geodesics

$$\mathcal{L} = g_{ab}\,\dot{x}^a \dot{x}^b \qquad (\text{set } G = 1)$$

$$= -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2$$

$$+ r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)$$

Conserved quantities

1) $\mathcal{L}$ independent of t $\Rightarrow \dfrac{d}{d\lambda}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{t}}\right) = 0$

$$E := \left(1 - \frac{2M}{r}\right)\dot{t}$$

2) $\mathcal{L}$ independent of ϕ :

$$J = r^2 \sin^2\theta\,\dot{\phi}^2$$

3) $\mathcal{L}$ itself is conserved

$$-K = \mathcal{L} = \begin{cases} -1 & \text{TL geodesic} \\ 0 & \text{null geodesic} \end{cases}$$

In TL case, choose proper time as parameter.

Spherical symmetry $\Rightarrow (J_x, J_y, J_z)$ conserved

- J corresponds to ϕ rotations
- Other components are messy

Simplification: Can always rotate coordinates so particles initial motion is in $\theta = \frac{\pi}{2}$ plane, WLOG.

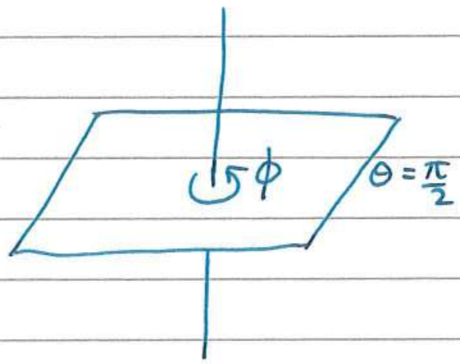
Particle motion stays in this plane

Proof: θ eq^{n} $\quad \frac{d}{d\lambda}\left(r^2\dot{\theta}\right) = r^2 \sin\theta\cos\theta\,\dot{\phi}^2$

$$\Rightarrow \quad \ddot{\theta} + \frac{2\dot{r}\dot{\theta}}{r} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0 \quad \circledast$$

Use symmetry to set

$$\theta(0) = \frac{\pi}{2}, \qquad \dot{\theta}(0) = 0$$



Then $\circledast$ implies $\ddot{\theta}(0) = 0$

$$\Rightarrow \quad \theta(s) = \frac{\pi}{2}$$

Then the only non-zero angular momentum is J.

Restrict to $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$

$$E = \left(1 - \frac{2M}{r}\right)\dot{t}$$

$$J = r^2\dot{\phi}$$

$$\mathcal{L} = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2$$

$$= -K$$

Combine (eliminate $\dot{t}$ and $\dot{\phi}$ from $\mathcal{L}$)

$$-K = \frac{\dot{r}^2 - E^2}{1 - \frac{2M}{r}} + \frac{J^2}{r^2}$$

$$\Rightarrow \quad E^2 = \dot{r}^2 + \left(K + \frac{J^2}{r^2}\right)\left(1 - \frac{2M}{r}\right)$$

$$\Rightarrow \quad \frac{E^2 - K}{2} = \frac{1}{2}\dot{r}^2 + V(r)$$

$$V(r) = \frac{-KM}{r} + \frac{J^2}{2r^2} - \frac{mJ^2}{r^3}$$

"Motion of massive particle in 1d with effective potential $V(r)$"

11.2 - Timelike Geodesics

TL $\Rightarrow$ $K = 1$

$$\Rightarrow \quad \frac{E^2 - 1}{2} = \frac{1}{2}\dot{r}^2 + V(r)$$

$$V(r) = \frac{-M}{r} + \frac{J^2}{2r^2} - \frac{mJ^2}{r^3}$$

Newtonian gravitational potential

Centrifugal barrier

GR correction

Compare to energy of non-relativistic particle in $1/r$ potential:

$$\mathcal{L} = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 + \frac{mM}{r}$$

$$\mathcal{H} = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 - \frac{M}{r}$$

$$= \frac{1}{2}\dot{r}^2 + \underbrace{\frac{J^2}{2r^2} - \frac{M}{r}}_{V_N(r)}$$

i.e. $$V(r) = V_N(r) - \frac{mJ^2}{r^3}$$

Properties of $V(r)$

- $V(r) \to -\frac{M}{r}$ as $r \to \infty$
- $V(r) \to -\frac{mJ^2}{R^3}$ as $r \to 0$

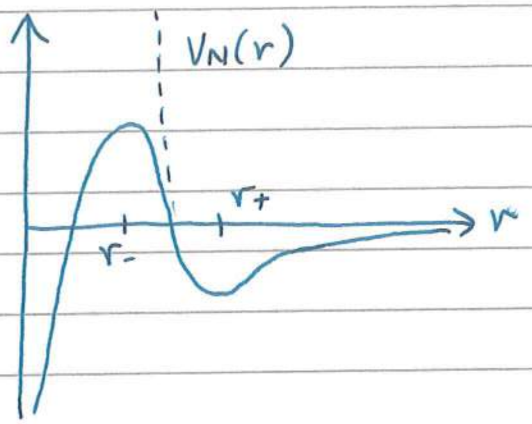
Extrema : $$\frac{\partial V}{\partial r} = \frac{M}{r^2} - \frac{J^2}{r^3} + \frac{3mJ^2}{r^4}$$

$$V' = 0 \quad \text{at} \quad r_\pm = \frac{J^2}{2M}\left(1 \pm \sqrt{1 - 12\left(\frac{M}{J}\right)^2}\right)$$

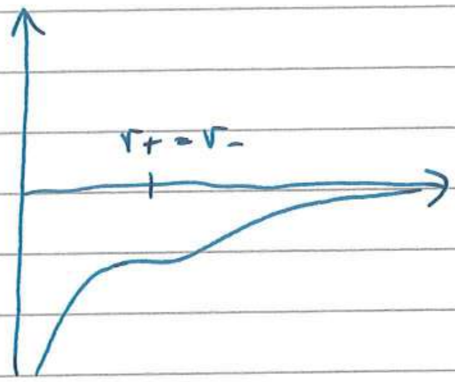
At $J/M = \sqrt{12}$, extrema collide to leave inflection point.

$J/M > \sqrt{12}$

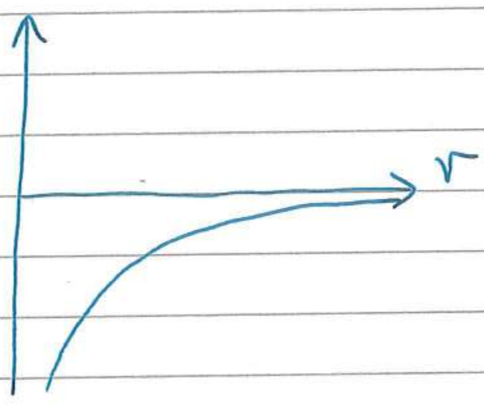
r_+ is minimum.



$J/M = \sqrt{12}$



$J/M < \sqrt{12}$



For this choice, the GR correction completely overcomes the angular momentum barrier ~~for~~ from the centrifugal term.

"Gravity in GR stronger than Newtonian"

11.3 - Types of Orbit

Recall : $\frac{E^2-1}{2} = \frac{1}{2}\dot{r}^2 + V(r)$

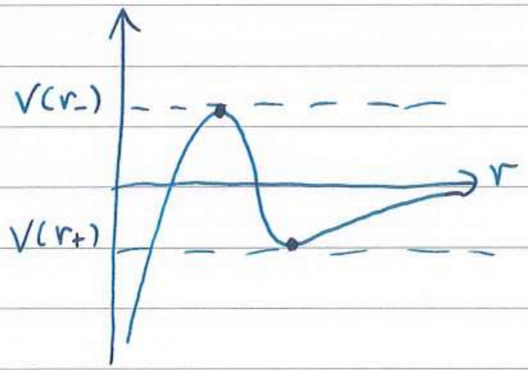
$\Rightarrow \ddot{r} = -\frac{\partial V}{\partial r}$ (can also get from r eq$^{\underline{n}}$)

a) Circular orbit

$\dot{r} = \ddot{r} = 0$

$\ddot{r} = 0 \Rightarrow \frac{\partial V}{\partial r} = 0$

which is true for $r = r_\pm$



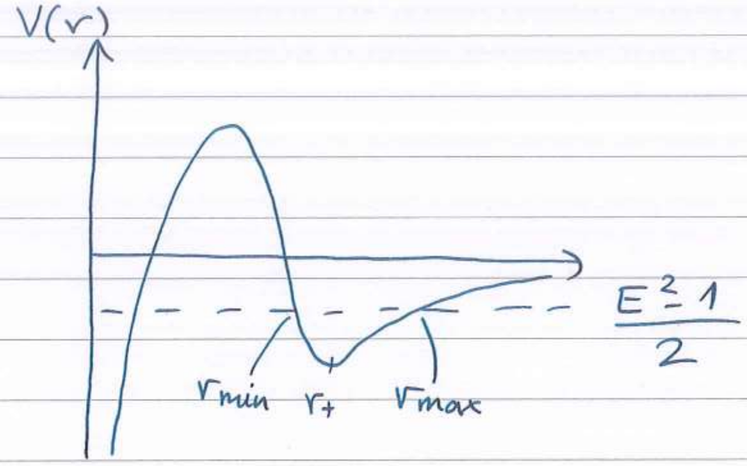
$r_+ > r_-$

- r_- is unstable orbit
- r_+ is stable orbit

The closest (marginally) stable orbit is for $J/M = \sqrt{12}$

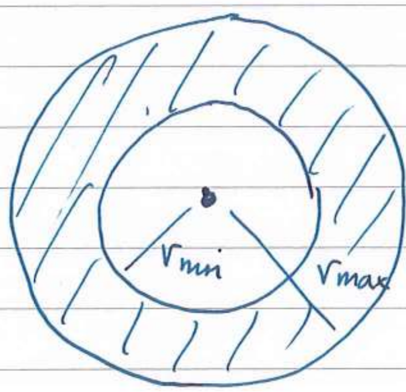
$r_\pm = \frac{J^2}{2M} = 6M$

b) Bound Orbits



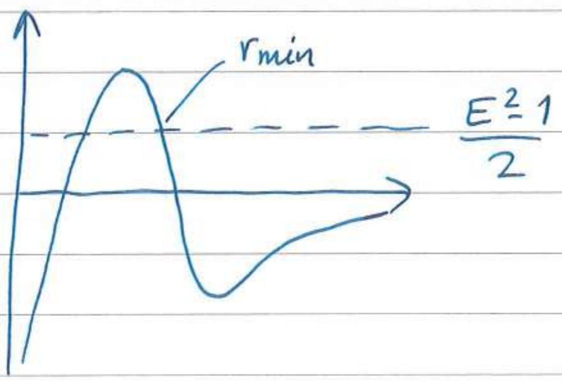
$$V(r_+) < \frac{E^2-1}{2} < 0$$

where $r > r_-$

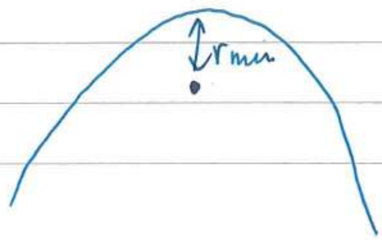


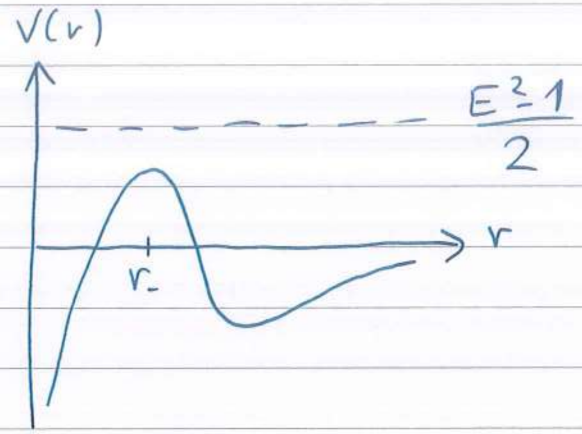
Orbit confined to shaded region.

c) Unbound orbits

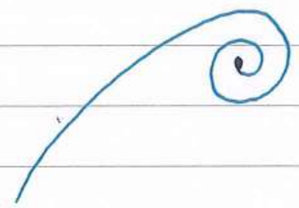


$$0 < \frac{E^2-1}{2} < V(r_-)$$





$$\frac{E^2-1}{2} > V(r_-)$$



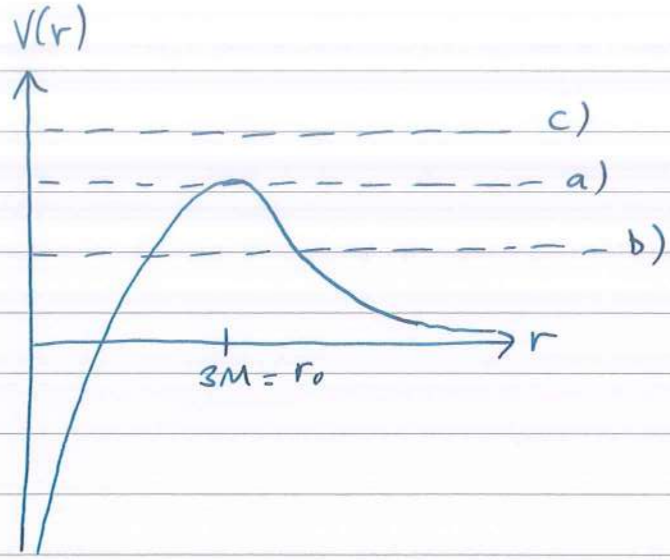
Such an orbit is not possible in Newtonian theory as angular momentum barrier is infinitely high.

11.4 - Null geodesics

$$k = 0 \Rightarrow \frac{E^2}{2} = \frac{1}{2}\dot{r}^2 + V(r)$$

$$V(r) = \frac{J^2}{2r^2} - \frac{MJ^2}{r^3}$$

No Newtonian potential term as have ~~massive~~ massless particles

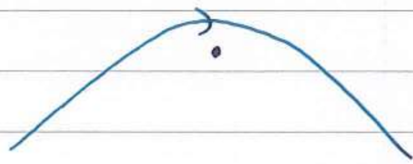


a) Unstable circular orbit

$r_0 = 3M \;, \qquad \frac{E^2}{2} = V(r_0)$

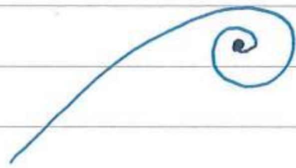
b) Unbound orbit

$$0 < \frac{E^2}{2} < V(r_0)$$



c) Unbound orbit

$$\frac{E^2}{2} > V(r_0)$$

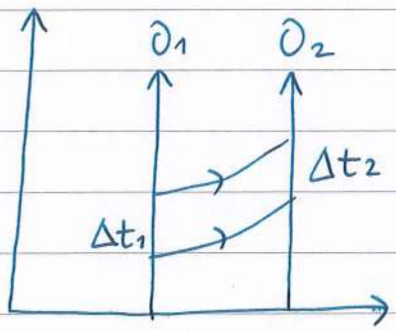


12.1 - Experimental Tests

- Gravitational redshift
- Perihelion shift
- Bending of light

Gravitational Redshift

(See lecture 10)



$$\Delta t_1 = \Delta t_2$$

$$\frac{\Delta \tau_1}{\Delta \tau_2} = \left(\frac{1 - \frac{2M}{r_1}}{1 - \frac{2M}{r_2}} \right)^{1/2}$$

If curves represent the crests of an EM wave, the <u>energies</u> measured by O_1 and O_2 are

$$E_1 = \frac{2\pi\hbar}{\Delta \tau_1} , \qquad E_2 = \frac{2\pi\hbar}{\Delta \tau_2}$$

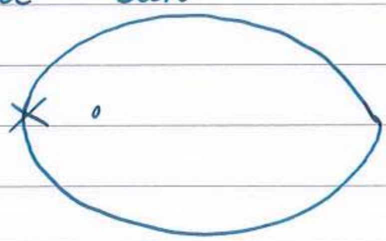
$$\frac{E_2}{E_1} = \left(\frac{1 - \frac{2M}{r_1}}{1 - \frac{2M}{r_2}} \right)^{1/2}$$

If $r_2 > r_1$, $\frac{E_2}{E_1} < 1$

Photons redshifted as they climb out of a gravitational potential.

12.3 – Perihelion Shift

Perihelion is point of closest approach of orbit around sun



Recall that for TL geodesic in Schwarzschild

Observation: perihelion shifts with each orbit. Newton predicts ~~2~~ 0 of the shift observed (43 arcseconds / century)

$$V(r) = -\frac{M}{r} + \frac{J^2}{2r^2} - \frac{MJ^2}{r^3}, \quad \frac{E^2-1}{2} = \frac{1}{2}\dot{r}^2 + V(r)$$

Consider a circular orbit at $r = r_*$ with small perturbations

$$0 > \frac{E^2 - 1}{2} > V(r_+) \quad ,$$

Step 1 :

Circular $\Rightarrow \ddot{r} = 0 = \frac{\partial V}{\partial r}$

$$\Rightarrow J^2 = \frac{Mr^2}{r - 3M}$$

For $r \gg M$, $J^2 \simeq Mr$

Step 2 : small perturbations around orbit.

Convenient to change $(r, \tau) \mapsto (u, \phi)$

$$u \equiv \frac{M}{r}$$

$$\frac{du}{d\phi} = -\frac{M}{r^2}\frac{dr}{d\phi}$$

$$= -\frac{M}{r^2}\frac{\dot{r}}{\dot{\phi}}$$

$$= -\frac{M\dot{r}}{J}$$

Rewrite equation relating conserved quantities

$$\frac{E^2-1}{2} = \frac{1}{2}\left(\frac{J}{M}\frac{du}{d\phi}\right)^2 - u + \frac{J^2}{2M^2}u^2 - \frac{J^2u^3}{M^2}$$

Take 2nd derivative

$$\frac{d^2u}{d\phi^2} = \frac{du}{d\phi}\frac{d}{du}\left(\frac{du}{d\phi}\right)$$

$$= \frac{1}{2}\frac{d}{du}\left(\frac{du}{d\phi}\right)^2$$

$$= \underbrace{\frac{M^2}{J^2} - u}_{\text{Newton}} + \underbrace{3u^2}_{\text{GR}}$$

<u>Perturb</u>: $u(\phi) = \frac{M}{R} + v(\phi)$

radius of circular orbit

$$\Rightarrow \frac{d^2v}{d\phi^2} = \frac{M^2}{J^2} - \left(\frac{M}{R}+v\right) + 3\left(\frac{M}{R}+v\right)^3$$

$$= \frac{M^2}{J^2} - \frac{M}{R} + 3\left(\frac{M}{R}\right)^2$$

$$- \left(1 - \frac{6M}{R}\right)v + \mathcal{O}(v^2)$$

From Step 1: $J^2 = \frac{Mr^2}{r-3M}$ the $\mathcal{O}(v^0)$ term vanishes

So $$\frac{d^2 v}{d\phi^2} + \left(1 - \frac{6M}{R}\right) v = 0$$

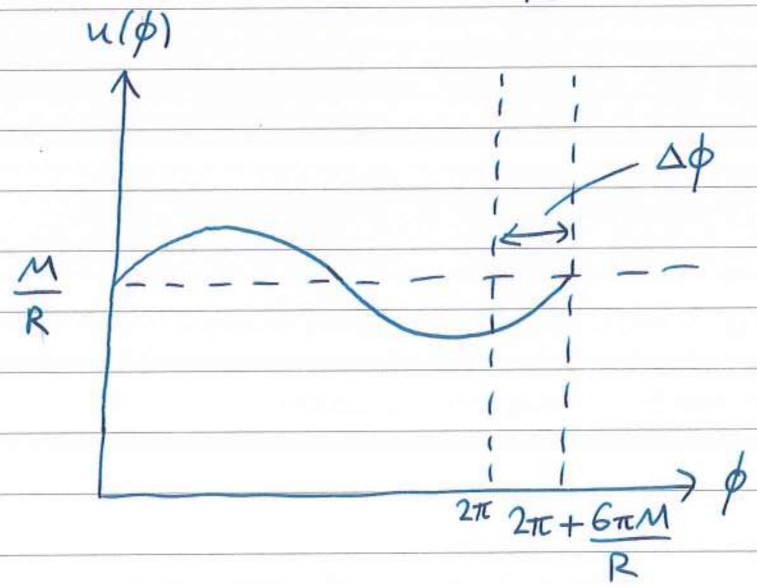
- $R < 6M$: unstable (hyperbolic sol$^{\underline{n}}$)
- $R > 6M$: stable (trig sol$^{\underline{n}}$)

For $R > 6M$, solution is periodic with frequency

$$\omega^2 = 1 - \frac{6M}{R}$$

$$T_\phi = \frac{2\pi}{\omega} = \frac{2\pi}{\left(1 - \frac{6M}{R}\right)^{1/2}}$$

$$\simeq 2\pi + \frac{6\pi M}{R} \quad \text{for } R \gg 3M$$



~~Orbit~~ Point of closest approach shifts by $\frac{6\pi M}{R}$ on each orbit!

Restoring units: $\Delta\phi = \frac{6\pi G M}{c^2 R}$

For Mercury: $R = 5.55 \times 10^7$ km

M = mass of sun = 1.99×10^{30} kg

$\Delta\phi \approx 5 \times 10^{-7}$ radians per orbit.

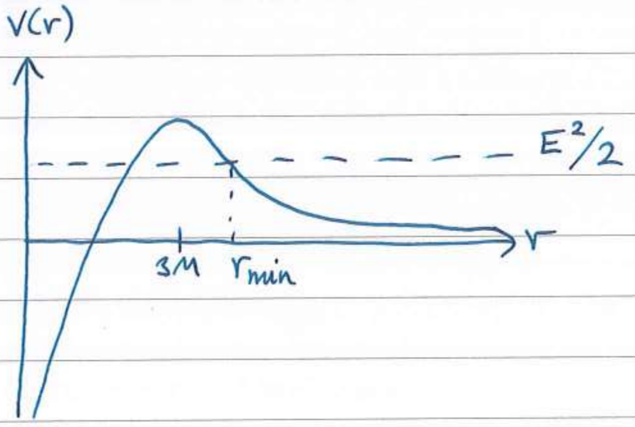
Matches observation!

12.4 - Bending of light.

$$\mathcal{L} = -K = 0$$

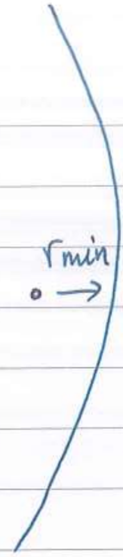
$$\Rightarrow \frac{E^2}{2} = \frac{1}{2}\dot{r}^2 + V(r)$$

$$V(r) = \frac{J^2}{2r^2}\left(1 - \frac{2M}{r}\right)$$



In region $0 < \frac{E^2}{2} < V(3M)$

$$E > \frac{J}{3M}$$



Again : $u = \frac{M}{r}$

$$\Rightarrow \frac{d^2u}{d\phi^2} = \underbrace{-u}_{\text{Newtonian}} + \underbrace{3u^2}_{\text{GR}}$$

For $r \gg M$, $u \ll 1$ and to leading order

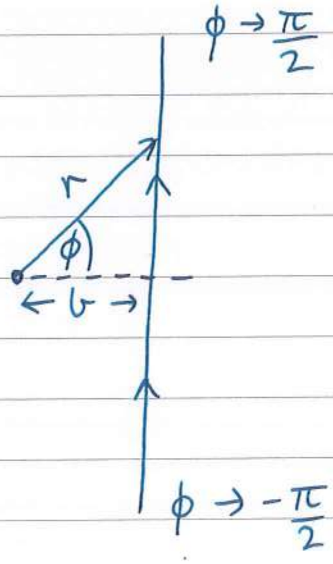
$$\frac{d^2u}{d\phi^2} = -u$$

$$\Rightarrow u(\phi) = A\cos(\phi - \phi_0)$$

Choose $\phi_0 = 0$

Substitute into $\left(\frac{ME}{J}\right)^2 = \left(\frac{du}{d\phi}\right)^2 + u^2$

$$\Rightarrow A = \frac{E}{J}M$$



"<u>Impact parameter</u>"

$$b = r\cos\phi$$

$$= \frac{M}{u}\cos\phi$$

$$= \frac{J}{E}$$

No deflection!

Now look at perturbation and include $\mathcal{O}(u^2)$ term.

$$u(\phi) = A\cos\phi + v(\phi)$$

with limit : $A \sim \mathcal{O}(\epsilon)$

$$v \sim \mathcal{O}(\epsilon^2)$$

$$\epsilon \to 0$$

A large impact parameter and a small perturbation. $v \ll A$.

Look at eq$^{\underline{n}}$ for $\left(\frac{du}{d\phi}\right)^2$

$$0 = \left(\frac{du}{d\phi}\right)^2 + u^2 - 2u^3 - A^2$$

$$= \left(-A\sin\phi + v'\right)^2 + \left(A\cos\phi + v\right)^2$$

$$- 2\left(A\cos\phi + v\right)^3 - A^2$$

- Leading $\mathcal{O}(\epsilon^2)$ term vanishes by construction
- Subleading terms give

$$0 = -2A\sin\phi\, v' + 2A\cos\phi\, v - 2A^3\cos^3\phi$$

$$\Rightarrow \quad \sin\phi\, v' = \cos\phi\, v - A^2\cos^3\phi$$

$$\Rightarrow \quad v(\phi) = A^2(1+\sin^2\phi) + C\sin\phi$$

Choose $v(-\frac{\pi}{2}) = 0$ (no perturbation at ∞)

$$= 2A^2 - C$$

$$\Rightarrow v(\phi) = A^2(1+\sin\phi)^2$$

Deflection angle $\Delta\phi$ is $r \to \infty$ for $\phi > 0$

$$0 = u(\tfrac{\pi}{2} + \Delta\phi)$$

$$= 4A^2 - A\Delta\phi + \mathcal{O}(\Delta\phi^2)$$

$$\Rightarrow \Delta\phi = 4A$$

$$= \frac{4ME}{J}$$

$$= \frac{4M}{b}$$

Restoring units : $\Delta\phi = \dfrac{4GM}{bc^2}$

- Deflection larger for greater mass and closer approach.

13.1 - Singularities

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2$$
$$+ r^2 d\Omega^2$$

What happens at $r = 2M, 0$?

Singularities can be artifact of coordinates

Compute scalar invariants:

- $R \equiv g^{ab} R_{ab} = 0$ (as solves Einstein)
- $R_{abcd} R^{abcd} = \frac{12M^2}{r^6}$

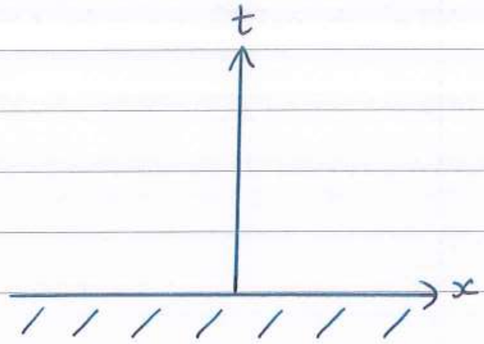
$\hookrightarrow$ $r = 0$ is genuine singularity in geometry

Nothing happens to R^2 at $r = 2M$. Singularity in metric components can be removed by a coordinate transformation

$r = 2M$ plays important physical role, known as "event horizon".

13.2 - Toy Example I

$$ds^2 = -\frac{dt^2}{t^4} + dx^2$$



$$0 < t < \infty, \quad -\infty < x < \infty$$

Singularity in metric components at $t = 0$

Change coordinates: $t' = t^{-1}$

$$dt' = -\frac{dt}{t^2}$$

$$\Rightarrow \quad ds^2 = -dt'^2 + dx^2$$

Metric on half of flat $\mathbb{R}^2$ with $0 < t' < \infty$

Singularity at $t = 0 \equiv t' \to \infty$ region in flat $\mathbb{R}^2$.

<u>Definition</u>: Spacetime is <u>geodesically complete</u> if all geodesics can be extended to arbitrarily ~~large~~ values of affine parameter.

(t', x) coordinates show geodesics can extend out to $t' \to \infty$ $(t = 0)$ without issue

- Spacetime <u>not</u> geodesically complete as $t' = 0$ is limit.

Can <u>extend</u> spacetime so it is geodesically complete.

- In (t, x), extend past $t \to \infty$
- In (t', x), just include $t' \leq 0$ to get full $\mathbb{R}^2$.

13.3 – Toy Example II

Rindler spacetime

$$ds^2 = -x^2 dt^2 + dx^2$$

$$-\infty < t < \infty$$
$$0 < x < \infty$$

Metric components singular at $x = 0$

$$g_{ab} = \begin{pmatrix} -x^2 & \\ & 1 \end{pmatrix} \qquad g^{ab} = \begin{pmatrix} -1/x^2 & \\ & 0 \end{pmatrix}$$

- Geodesics terminate with finite length at $x = 0$
- Curvature invariants not singular at $x = 0$ ($R_{abcd} = 0$!)

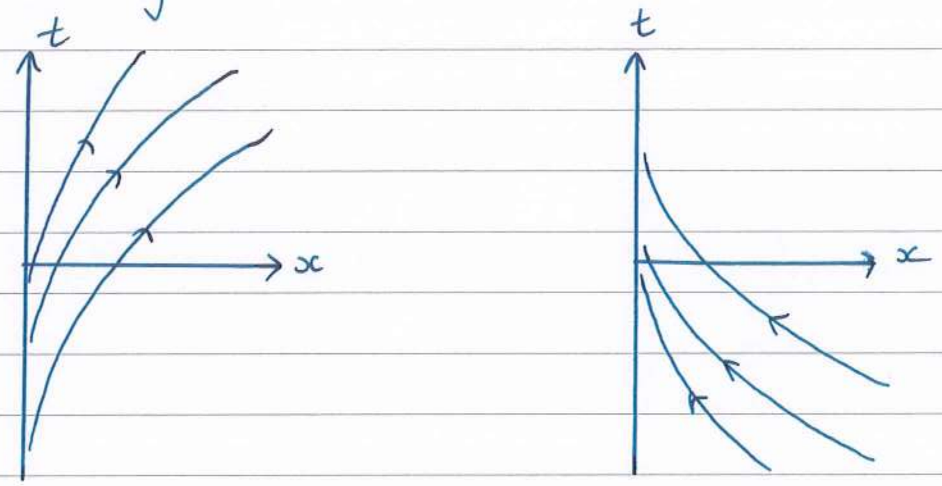
Introduce coordinates using <u>null</u> geodesics

$$\mathcal{L} = -x^2 \dot{t}^2 + \dot{x}^2 = 0$$

$$\Rightarrow \left(\frac{dt}{dx}\right)^2 = \frac{1}{x^2}$$

$$\Rightarrow \quad t = \pm \log x + \text{const.}$$

$+$: outgoing
$-$: incoming



Define "<u>null coordinates</u>" (u, v)

$$u = t - \log x$$

$$v = t + \log x$$

Incoming null geodesic $\Rightarrow$ $v = \text{const.}$

Outgoing null geodesic $\Rightarrow$ $u = \text{const.}$

Compute metric in (u, v) coordinates

$$du = dt - \frac{dx}{x}$$

$$dv = dt + \frac{dx}{x}$$

$$\Rightarrow \quad du\,dv = dt^2 - \frac{dx^2}{x^2}$$

$$\Rightarrow \quad ds^2 = -e^{v-u}\,du\,dv$$

where $-\infty < u < \infty$ and $-\infty < v < \infty$ corresponds to $x > 0$

Note, since $x^2 = e^{v-u}$, $x = 0$ corresponds to $v = -\infty$ or $u = +\infty$.

Is the space geodesically complete? Look near $x = 0$.

Let's compute the affine parameter λ along null geodesics.

$$\frac{\partial}{\partial t} L = 0 \quad \Rightarrow \quad E = x^2 \dot{t} = \text{const.}$$

$$= x^2 \frac{dt}{d\lambda}$$

$$\Rightarrow \quad d\lambda = \frac{1}{E} x^2\,dt$$

$$= \frac{1}{2E} e^{v-u}(du + dv)$$

Outgoing : $u = u_0$ (constant)

$$\Rightarrow \quad \lambda = \frac{1}{2E} \int e^{v - u_0} \, dv$$

$$= C + \frac{e^{-u_0}}{2E} e^{v}$$

$$\equiv a + b\lambda'$$

i.e. e^{v} is affine parameter

Incoming : $v = v_0$

$$\Rightarrow \lambda = \frac{1}{2E} \int e^{-u + v_0} \, du$$

$$= C - \frac{e^{v_0}}{2E} e^{-u}$$

i.e. $-e^{-u}$ is affine parameter

Can we follow geodesics for any length?

i.e. do affine parameters take values in all of $\mathbb{R}$?

No! $\quad \lambda_{in} = e^{v}, \quad \lambda_{out} = -e^{-u}$

$$\lambda_{in} \geq 0 , \quad \lambda_{out} \leq 0$$

<u>Idea</u> : change coordinates so that metric has no singularity at $x = 0$, then just extend range of coordinates to give new, extended spacetime.

Let $U = -e^{-u}$, $V = e^{v}$, $U \in (-\infty, 0)$ $V \in (0, \infty)$

$$\Rightarrow \quad ds^2 = -dU\,dV$$

No singularity at $U = V = 0$ $(x = 0)$

- Extend to $U \in (-\infty, \infty)$ $V \in (-\infty, \infty)$

- This extended spacetime is geodecically complete (cannot access new region with old (u, v) coords.)

Finally, set $T = \frac{1}{2}(U + V)$

$$X = \frac{1}{2}(V - U)$$

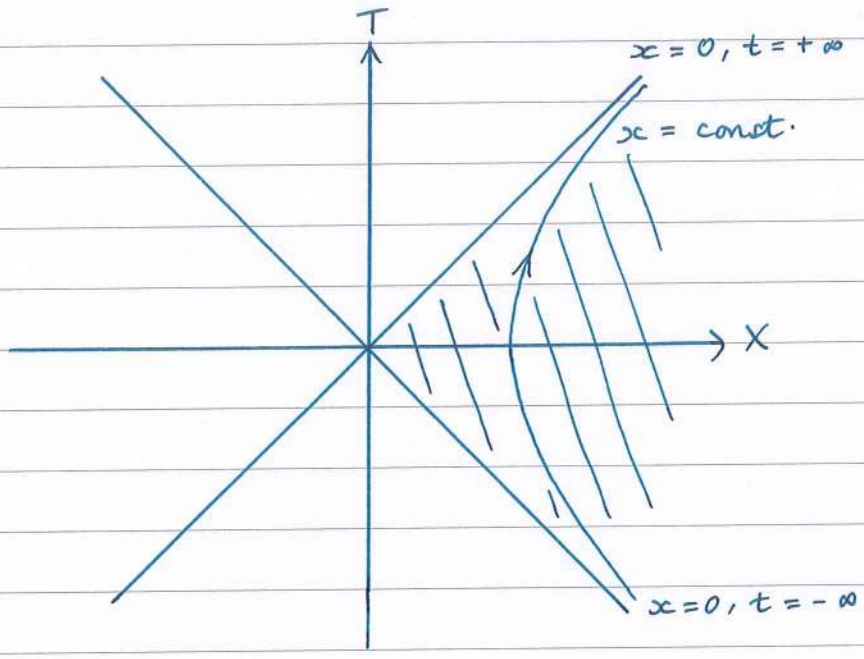
$$\Rightarrow \quad ds^2 = -dT^2 + dX^2$$

$T \in (-\infty, \infty)$, $X \in (-\infty, \infty)$

Simply $\mathbb{R}^2$ with <u>flat</u> metric!

If there were a curvature singularity at $x = 0$, no reason to extend spacetime!

- Original spacetime is region $X > |T|$
- Null geodesics are <u>straight lines</u>.
- Curve $x = \text{const.}$ is uniformly acc. observer in (T, X) coordinates



13.4 - Back to Schwarzschild

In $r > 2M$ region

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Apply same method: look at null geodesics, find coordinates well-behaved at $r = 2M$, extend coordinates.

<u>Null geodesics</u>: $\mathcal{L} = 0 = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)\dot{r}^2$

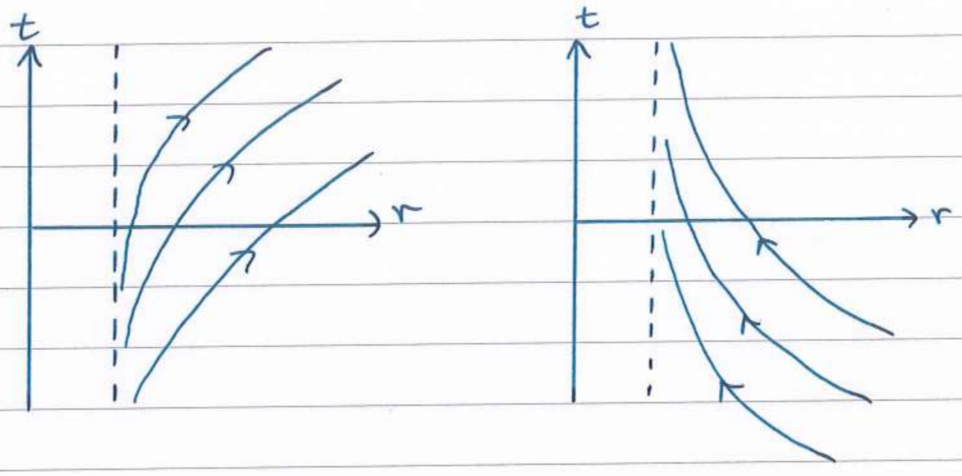
$$\Rightarrow \left(\frac{dt}{dr}\right)^2 = \left(\frac{1}{1 - \frac{2M}{r}}\right)^2$$

$$\Rightarrow \quad t = \pm r_* + \text{const.}$$

where $r_* = r + 2M \log\left(\frac{r}{2M} - 1\right)$

Out

$+$



In

$-$

Use null coordinates in $r > 2M$

$$u = t - r_*$$

$$v = t + r_*$$

<u>Incoming</u>: $v = v_0$

<u>Outgoing</u>: $u = u_0$

<u>Metric</u>: $ds^2 = -\left(1 - \frac{2M}{r}\right) du\, dv$

$$= -\frac{2M}{r} e^{-r/2M} e^{(v-u)/4M} du\, dv$$

with $\quad u \in (-\infty, \infty), \quad v \in (-\infty, \infty)$

$r \to 2M$ corresponds to $u \to +\infty$ or $v \to -\infty$

This is not geodesically complete : radial null geodesics meet $r = 2M$ at finite affine parameter

c.f. $\quad \left(1 - \frac{2M}{r}\right)\dot{t} = \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}$ for null

$$\Rightarrow \quad dt = \pm \frac{dr}{1 - \frac{2M}{r}}$$

but $\quad E = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\lambda}$

$$\Rightarrow \quad d\lambda = \pm \frac{1}{E} dr$$ so r is affine.

Define : $\quad U = -e^{-u/4M} \in (-\infty, 0)$

$$V = e^{v/4M} \in (0, \infty)$$

$$\Rightarrow \quad ds^2 = -\frac{32M^3}{r} e^{-r/2M} \, dU \, dV$$

Metric is non-singular at $r = 2M$ ($U = 0$ or $V = 0$) so we extend to

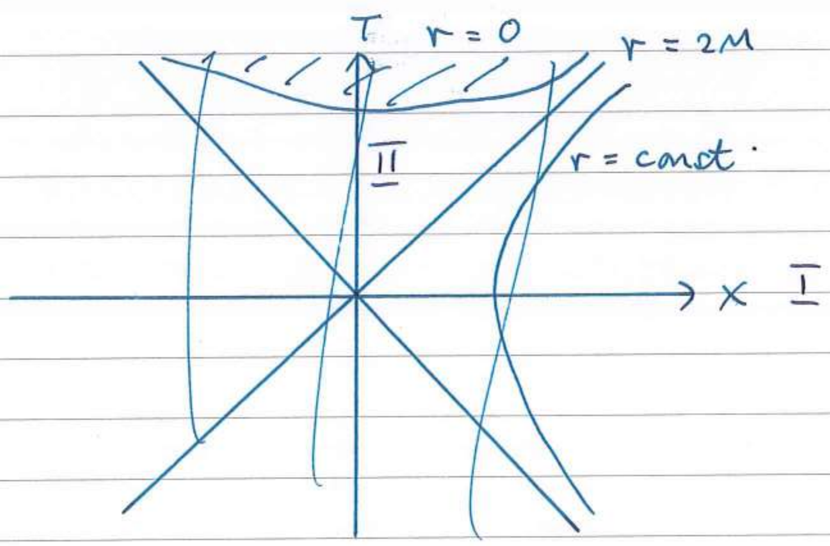
$$U \in (-\infty, \infty), \quad V \in (-\infty, \infty)$$

Finally : $T = \frac{1}{2}(U+V)$

$X = \frac{1}{2}(V-U)$

- $r > 0$ requires $T^2 - X^2 < 1$
- Metric : $ds^2 = \frac{32m^2}{r} e^{-r/2m}(-dT^2 + dX^2)$
- Null geodesics : incoming $T + X = \text{const.}$

  outgoing $T - X = \text{const}$
- $r > 2M$ region is $X > |T|$
- Now nothing strange happens at $r = 2M$. Can follow a particle down to $r = 2M$ using (t, r), then follow across horizon with with (U, V), then convert back to (t, r) inside horizon.



Consider observer falling across horizon

For $r < 2M$

$$ds^2 = + \underbrace{\left(\frac{2M}{r} - 1\right)}_{>0} dt^2 - \underbrace{\left(\frac{2M}{r} - 1\right)^{-1}}_{>0} dr^2$$

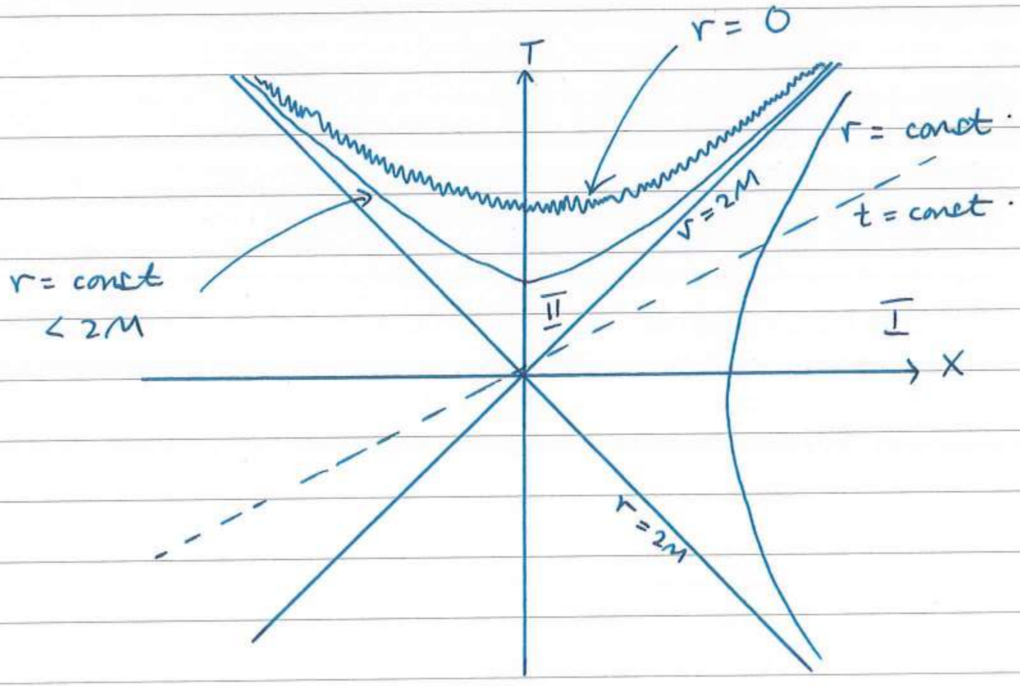
t is now a <u>spacelike</u> coordinate.

r is <u>timelike</u> coordinate.

~~For a timelike geodesic~~

- r decreases as particles move to the future

- $r = 0$ (where $R_{abcd} R^{abcd}$ diverges) is a point in time. It cannot be avoided.



- Particle can never escape from $r < 2M$
- Will reach $r = 0$ in finite proper time
- Signals sent from II always hit $r = 0$

"Black Hole"

14.1 - Cosmology

Cosmology is large-scale description of universe.

- Cannot do easy "cosmological experiments" so there are many things we don't understand

- What happened at (or before) Big Bang?

- Is universe spatially finite or infinite?

- Will universe continue expanding?

- What is dark matter?

- Is dark energy causing acceleration? What is dark energy?

Build models using observations

- Impossible to write down models for all stars etc.

- Average over galactic scales to get an approximate model.

1) At large scales, non-gravitational interactions can be ignored (short range or screened).

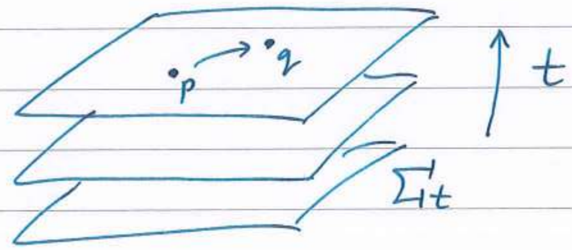
2)

2) No privileged position in the universe - "Copernican Principle". Universe looks the same everywhere, i.e. space is homogeneous.

3) Universe looks the same in all directions. Rotational symmetry, i.e. space is isotropic.

14.2 - Homogeneity

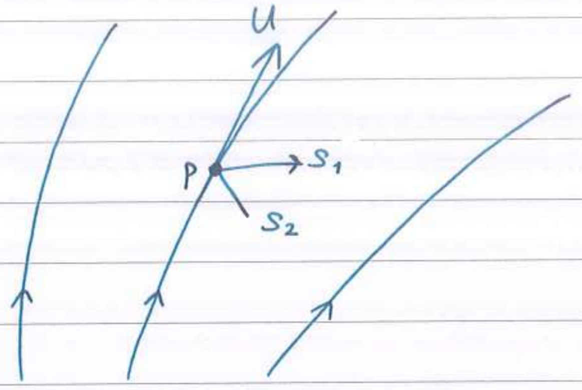
Spacetime foliated by 1-parameter family of spacelike hypersurfaces



For any 2 points $p, q \in \Sigma_t$, there is a transformation $p \mapsto q$ that preserves the metric - an "isometry".

14.3 - Isotropy

There is a set of distinguished observers (timelike geodesics) filling spacetime such that



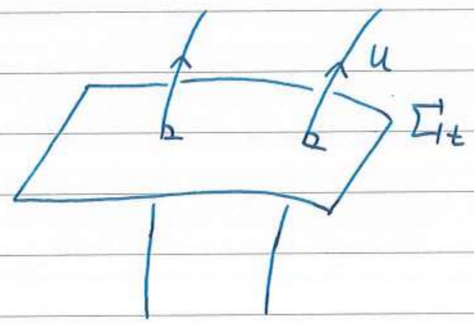
$$g(u, s_1) = g(u, s_2) = 0$$

for any point p and a pair of spacelike vectors orthogonal to u, there exists an isometry of the metric that rotates $s_1 \mapsto s_2$.

Known as "isotropic observers".

13.4 - Metric

Homog. + Iso. $\Rightarrow$ Σ_t orthogonal to u



$\Rightarrow \quad g_{ab} = u_a u_b + h_{ab}$ $\longrightarrow$ metric on Σ_t

Introduce coordinates : $x^a = (\tau, x^i)$

(co-moving)
τ = proper time of <u>isotropic</u> observers

(homog $\Rightarrow$ isotropic observers measure same $\Delta\tau$ between hypersurfaces)

In these coordinates

$$ds^2 = -d\tau^2 + h_{ij}\, dx^i dx^j$$

Let $R_{ij}{}^{k}{}_{l}$ be the Riemann tensor on Σ_τ built from h_{ij}

<u>Claim</u>: 1) Isotropy $\Rightarrow$ $R_{ijkl} = \frac{R}{6}(g_{ik}g_{jl} - g_{jk}g_{il})$

2) Homog. $\Rightarrow$ R is constant on Σ_τ (depends on τ only)

(Proof of 1) at end of notes)

A space with R_{ijkl} of this form and R = const is a space of <u>constant curvature</u>.

Taken together, Σ_τ must be a <u>maximally symmetric</u> space. There are 3 of these

- <u>$R > 0$</u> : 3-sphere S^3 with round metric

$$ds^2_{(3)} = d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

- $\underline{R = 0}$: Flat $\mathbb{R}^3$ with Euclidean metric

$$ds^2_{(3)} = d\psi^2 + \psi^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

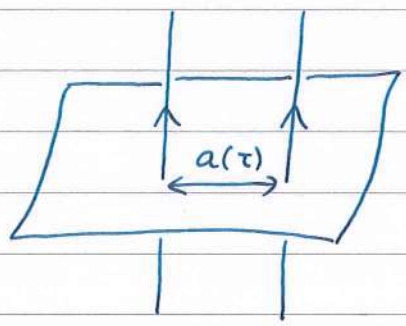
- $\underline{R < 0}$: Hyperbolic space H^3

$$ds^2_{(3)} = d\psi^2 + \sinh^2\psi \, (d\theta^2 + \sin^2\theta \, d\phi^2)$$

The 4d metric is then <u>Friedmann - Robertson - Walker</u>

$$ds^2 = -d\tau^2 + a^2(\tau) \, ds^2_{(3)}$$

The <u>scale factor</u> $a(\tau)$ gives separation of nearby isotropic observers



$a(\tau) > 0$ by definition.

To determine $a(\tau)$, solve Einstein equation given

- Choice of $R > 0$, $R = 0$ or $R < 0$.
- Choice of matter content.
- Choice of cosmological constant Λ (to come).

Comments

- Can also take

$$ds_{(3)}^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2$$

S^2

- Previous metric has $k=1$ for S^3, etc.

- In this metric, can use rescaling of $a(\tau)$ to set $k = \pm 1, 0$.

- Or use scaling to set $a(\tau_0) = 1$ where τ_0 = "today".

- At constant τ, $ds^2 = a(\tau)^2 ds_{(3)}^2$

  - For $k=1$, $a(\tau)$ gives size (radius) of universe.

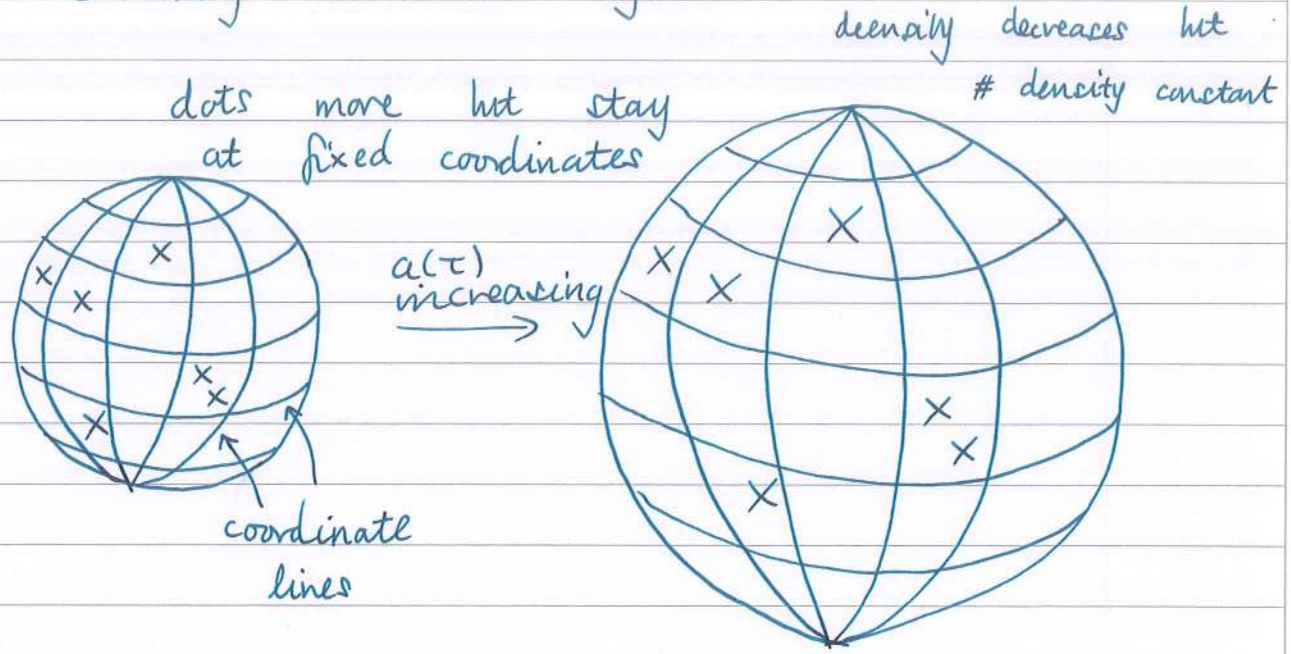
  - For $k=-1$, universe is infinite but $a(\tau)$ sets scale of curvature

$$R^{(3)}(\tau) = \frac{1}{a^2(\tau)} R^{(3)}$$

  - For $k=0$, $\mathbb{R}^3$ infinite.

NB: $k=1$ case gives option of unbounded finite universe (S^3 is boundary of ball in 4d).

e.g. Comoving coordinate system



One can check that x^i = constant curves are geodesics. (or ψ = constant)

Co-moving observers at ψ = constant follow geodesics.

Their proper time is just the τ that appears in the metric.

<u>Proof of $R_{ijkl} = \frac{R}{6}(g\ldots + \quad)$</u>

From $R_{ij}{}^{kl} = -R_{ji}{}^{kl} = -R_{ij}{}^{lk}$, we have a linear map on the vector space of ~~rank~~ (0,2) antisymmetric tensors at $p \in \Sigma_\tau$, W.

$$R : W \to W ,$$

$$: A_{ij} \to R_{ij}{}^{kl} A_{kl}$$

Since $R_{ij}{}^{kl} = + R^{kl}{}_{ij}$, the map is self-adjoint with respect to the +ve definite inner product on W

$$\langle -, - \rangle : W \times W \to \mathbb{R}$$

$$: (A_{ij}, B_{ij}) \mapsto h^{ik} h^{jl} A_{ij} B_{kl}$$

W is finite dimensional, so has an orthonormal basis of eigenvectors of $R : W \to W$.

If the eigenvalues were distinct, we could construct distinguished vectors which would violate isotropy.

Thus all eigenvalues must be equal

$\Rightarrow$ R = identity map.

$\Rightarrow$ $R = k\,\mathbb{1}$

i.e. $R_{ij}{}^{kl} = k(\delta_i{}^k \delta_j{}^l - \delta_j{}^k \delta_i{}^l)$

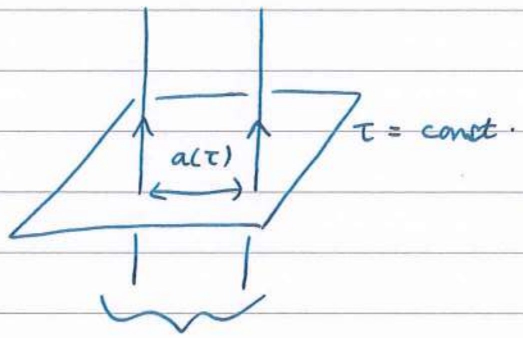
Contracting to get $R = 6k$

Homogeneity implies R cannot depend on x^i, so it is constant on Σ_t.

15.1 - Review

Homogeneity + Isotropy:

$$ds^2 = -d\tau^2 + a(\tau)^2 \begin{cases} d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi^2) \\ d\psi^2 + \psi^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\ d\psi^2 + \sinh^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi^2) \end{cases}$$



Isotropic / comoving observers with proper time τ and $u^a = (1, \underline{0})$

$a(\tau)$ is scale factor, fixed by Einstein equations.

15.2 - Curvature

Focus on <u>flat</u> spatial geometry $(k=0)$

$$ds^2 = -d\tau^2 + a^2\, dx^i dx^i$$

1) Compute Γ from $\mathcal{L}$

$$\mathcal{L} = -\dot{\tau}^2 + a^2 \dot{x}^i \dot{x}^i \quad , \quad \dot{\tau} = \frac{d\tau}{d\lambda} \quad , \quad a' = \frac{da}{d\tau}$$

$$\Rightarrow \quad \Gamma^{\tau}{}_{ii} = a a' \quad , \quad \Gamma^{i}{}_{i\tau} = \frac{a'}{a}$$

NB : $\Gamma^{\tau}{}_{ii}$ and $\Gamma^{i}{}_{i\tau}$ equal for any i due to isotropy.

2) Isotropy $\Rightarrow$ R_{aa} (no sum) are possibly non-zero.

Find $$R_{\tau\tau} = -\frac{3a''}{a}$$

$$R_{ii} = aa'' + 2(a')^2$$

3) $$R = g^{ab} R_{ab}$$

$$= -R_{\tau\tau} + \frac{1}{a^2} \sum_{i=1}^{3} R_{ii}$$

$$= \frac{3a''}{a} + \frac{1}{a^2} 3 \left(aa'' + 2(a')^2 \right)$$

$$= 6 \left(\frac{a''}{a} + \frac{(a')^2}{a^2} \right)$$

Recall $G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab}$

$$\Rightarrow \quad G_{\tau\tau} = R_{\tau\tau} + \frac{1}{2} R$$

$$= \frac{3(a')^2}{a^2}$$

$$\Rightarrow \quad G_{ii} = -2aa'' - (a')^2$$

15.3 - Matter

Need to specify matter content of universe

In general one finds

$$G_{00} = 3\left(\frac{(a')^2}{a^2} + \frac{k}{a^2}\right)$$

$$G_{ij} = -\left(\frac{2a''}{a} + \frac{(a')^2}{a^2} + \frac{k}{a^2}\right) \underset{\uparrow \; a^2 ds^2_{(3)}}{h_{ij}}$$

Given $G_{ab} = 8\pi T_{ab}$, we must have

$$T_{00} = \rho(\tau)$$

$$T_{ij} = p(\tau) \, h_{ij}$$

Written in terms of the 4-velocity of a comoving observer u_a

$$T_{ab} = (p+\rho) u_a u_b + p g_{ab}$$

i.e. <u>perfect fluid</u> at rest w.r.t. comoving coordinates.

Supplement with equation of state $p = w\rho$

e.g. <u>dust</u> : $w = 0 \quad (p = 0)$ $\qquad S = S(V^w E)$

<u>radiation</u> : $w = 1/3$

cosmological constant : $w = -1$

NB : $G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$

$$\Rightarrow \quad G_{ab} = 8\pi \left(T_{ab} - \frac{\Lambda}{8\pi} g_{ab} \right)$$

$$\Rightarrow \quad T^{\Lambda}_{ab} = -\frac{\Lambda}{8\pi} g_{ab}$$

$$\Rightarrow \quad T^{\Lambda}_{00} \equiv \rho_\Lambda = \frac{\Lambda}{8\pi} = -p_\Lambda$$

(can also have $w = w(\tau)$, e.g. inflation).

15.4 - Einstein equations

Focus on $k = 0$, $T_{00} = \rho$

$$T_{ii} = a^2 p$$

$G_{ab} = 8\pi T_{ab}$

$\hookrightarrow$ 1) $3\left(\frac{a'}{a}\right)^2 = 8\pi \rho$

2) $-2aa'' - (a')^2 = 8\pi a^2 p$

1) + 2) $\Rightarrow \quad \frac{a''}{a} = -\frac{4\pi}{3}(\rho + 3p)$

Conservation of T_{ab}

$$\nabla_a T^{ab} = 0 \quad \Rightarrow \quad \rho' + 3(\rho + p)\frac{a'}{a} = 0$$

(τ component)

For <u>any</u> geometry

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2}$$

$$\frac{a''}{a} = -\frac{4\pi}{3}(\rho + 3p)$$

$$\rho' + 3(\rho + p)\frac{a'}{a} = 0$$

(not independent, any 2 imply the 3rd)
c.f. Bianchi

<u>15.5 - Examples</u>

a) <u>Dust</u> : pressureless gas $p = 0$ (e.g. galaxies)

Continuity eq$^{\underline{n}}$: $\rho' + 3\rho\frac{a'}{a} = 0$

$$\Rightarrow \quad \frac{\rho'}{\rho} + \frac{3a'}{a} = 0$$

$$\Rightarrow \quad \rho(a) \propto a^{-3}$$

Note: - ρ is rest mass per unit volume measured by isotropic observers

- Rest mass conserved

- Spatial volumes scale as a^3

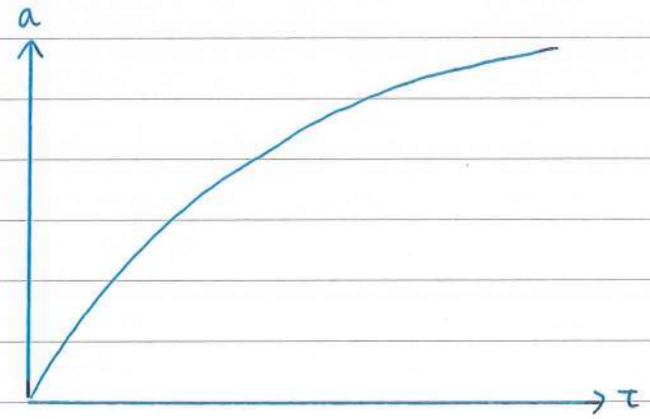
- expect $\rho \sim a^{-3}$

For $k = 0$

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi}{3}\rho = c\,a^{-3}$$

$$\Rightarrow \quad a' = \sqrt{c}\, a^{-1/2}$$

$$\Rightarrow \quad a(\tau) \propto \tau^{2/3}$$



v) <u>Radiation</u> filled universe, $p = \frac{1}{3}\rho$

Continuity: $\rho' + 4\rho \dfrac{a'}{a} = 0$

$$\Rightarrow \quad \rho(a) \propto a^{-4}$$

- a^{-3} from volume scaling
- a^{-1} from redshift of photons

For $k = 0$, $a(\tau) \propto \tau^{1/2}$

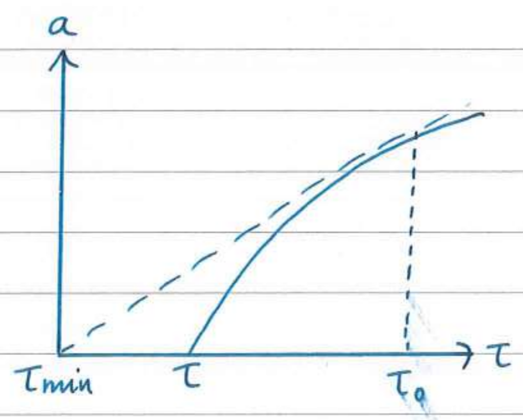
15.6 – General conclusions

From $\frac{a''}{a} = -\frac{4\pi}{3}(\rho + 3p)$ ($a > 0$ by definition)

if $\rho > 0$, $p > 0$, $\Rightarrow a'' < 0$

i.e. universe cannot be static!

Given $a' > 0$ today (expansion) and $a'' < 0$ (slowing expansion), we must have $a = 0$ at some point τ in the past



Given $a' > 0$, fate of universe determined by $k = \pm 1, 0$.

From continuity equation, if $p > 0$, ρ decreases at least as fast as a^{-3}

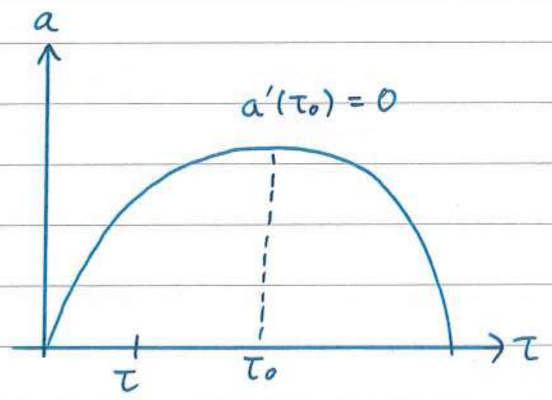
From $\left(\frac{a'}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2}$

<u>$k = 0, -1$</u>: If $a' > 0$ today, RHS $\neq 0$ so $a' > 0$ for all future

"Universe expands forever"

<u>$k = +1$</u>: If $a' > 0$ today, ρ decreases more rapidly than a^{-2} as a increases.

RHS $= 0$ for some τ, then $a' < 0$ for future.



Universe with compact spatial geometry (S^3) exists for finite time!

<u>NB</u>: Λ contributes $p < 0$ so these conclusions change depending on Λ!

16.1 - Cosmological metric:

Recall

$$ds^2 = -d\tau^2 + a(\tau)^2 \begin{cases} d\psi^2 + \sin^2\psi \, d\Omega^2 \\ d\psi^2 + \psi^2 \, d\Omega^2 \\ d\psi^2 + \sinh^2\psi \, d\Omega^2 \end{cases}$$

Scale factor $a(\tau)$ obeys

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2}$$

$$\frac{a''}{a} = -\frac{4\pi}{3}(p + 3\rho)$$

$$\rho' + 3(\rho + p)\frac{a'}{a} = 0$$

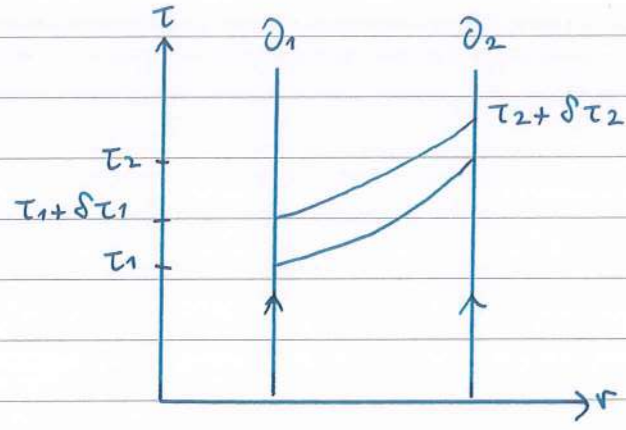
$$k = \begin{cases} +1 \\ 0 \\ -1 \end{cases}$$

Setting $r = \begin{cases} \sin\psi \\ \psi \\ \sinh\psi \end{cases}$ gives

$$ds^2 = -d\tau^2 + a^2(\tau)\left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right)$$

16.2 - Gravitational redshift

Two co-moving observers O_1 and O_2



Photon travels on null geodesic

$$\mathcal{L} = -\dot{\tau}^2 + a^2 \frac{\dot{r}^2}{1-kr^2} = 0$$

$$\Rightarrow \quad \frac{\dot{\tau}}{a} = \pm \frac{\dot{r}}{(1-kr^2)^{1/2}}$$

Take $+$ for $r_2 > r_2$ geodesic.

$$\Rightarrow \int_{\tau_1}^{\tau_2} \frac{d\tau}{a(\tau)} = \int_{r_1}^{r_2} \frac{dr}{(1-kr^2)^{1/2}}$$

RHS depends on (r_2, r_1) only

$$\Rightarrow \int_{\tau_1}^{\tau_2} \frac{d\tau}{a(\tau)} = \int_{\tau_1+\delta\tau_1}^{\tau_2+\delta\tau_2} \frac{d\tau}{a(\tau)}$$

$$\Rightarrow \int_{\tau_1}^{\tau_1+\delta\tau_1} \frac{d\tau}{a(\tau)} = \int_{\tau_2}^{\tau_2+\delta\tau_2} \frac{d\tau}{a(\tau)}$$

As $\delta\tau_1 , \delta\tau_2 \to 0$

$$\frac{\delta\tau_1}{\tau_1} = \frac{\delta\tau_2}{\tau_2}$$

O_1 emits photon with $\delta\tau_1$ between peaks,
O_2 measures it with $\delta\tau_2$ between peaks

$$E_i = \frac{2\pi\hbar}{\delta t_i}$$

$$\Rightarrow \frac{E_2}{E_1} = \frac{a(\tau_1)}{a(\tau_2)}$$

"Gravitational redshift in expanding universe"

$a(\tau_2) > a(\tau_1)$ for expansion $\Rightarrow$ $E_2 < E_1$

<u>NB</u>: Cosmologists write $\frac{E_2}{E_1} = \frac{1}{1+z}$

z known as the <u>redshift</u>.

16.3 - Hubble's law

Consider photon emitted from galaxy nearly at $\tau_1 = \tau_0 - \Delta\tau$ and received at $\tau_2 = \tau_0$

$$a(\tau_0 - \Delta\tau) \simeq a(\tau_0) - \Delta\tau\, a'(\tau_0) + \mathcal{O}(\Delta\tau^2)$$

But
$$z = \frac{a(\tau_2)}{a(\tau_1)} - 1$$

$$= \frac{a(\tau_0)}{a(\tau_0 - \Delta\tau)} - 1$$

$$= \Delta\tau \frac{a'(\tau_0)}{a(\tau_0)} + \mathcal{O}(\Delta\tau^2)$$

$$\simeq \Delta\tau\, H_0$$

$H_0 = \dfrac{a'(\tau_0)}{a(\tau_0)}$ is <u>Hubble's constant</u>

For $k = 0$, <u>physical distance</u> travelled by photon is

$$d = \int_{r_1}^{r_2} a(\tau_0)\, dr$$

"present instaneous proper distance"

$$\simeq a(\tau_0) \int_{\tau_0 - \Delta\tau}^{\tau_0} \frac{d\tau}{a(\tau)}$$

$$\approx \Delta\tau + \mathcal{O}(\Delta\tau^2)$$

$$\Rightarrow \quad z = H_0 d$$

"Hubble's law"

As $H_0 \sim$ constant, amount of redshift gives measure of distance to emitting objects

16 · 4 - Horizons

How much of the universe can be observed by a single co-moving observer?

Focus on $k = 0$. Study null geodesics.

Define conformal time

$$d\eta = \frac{d\tau}{a}$$

$$\eta = \int^{\tau} \frac{d\tau'}{a(\tau')}$$

Metric is then

$$ds^2 = a(\eta)^2 \left(-d\eta^2 + dx^i dx^i \right)$$

Flat metric on subset of $\mathbb{R}^4$ (might be horizons!) given by (η_{ab}, U), $U \subset \mathbb{R}^4$

This metric is conformally flat, light rays travel at 45° on spacetime diagrams.

The <u>causal structure</u> (how light behaves) is then easy to analyse.

Since $a^2 > 0$, events connected by a null geodesic iff connected by a geodesic in flat spacetime.

Need to worry about the range of η to see how $U \subset \mathbb{R}^4$.

If $\int^{\infty} \frac{d\tau}{a} \to \infty$ and $\int_{-\infty} \frac{d\tau}{a} \to -\infty$

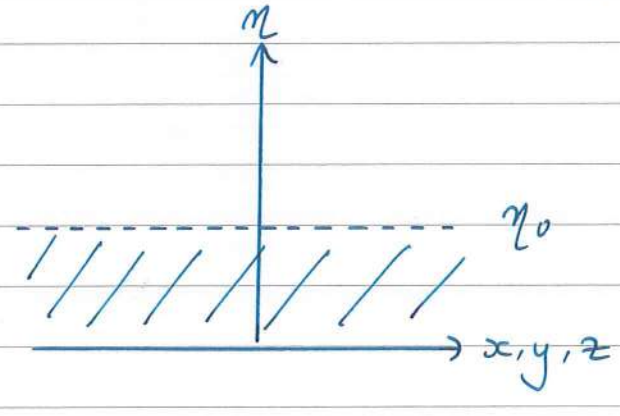
then $\eta \in (-\infty, \infty)$ so $U = \mathbb{R}^4$

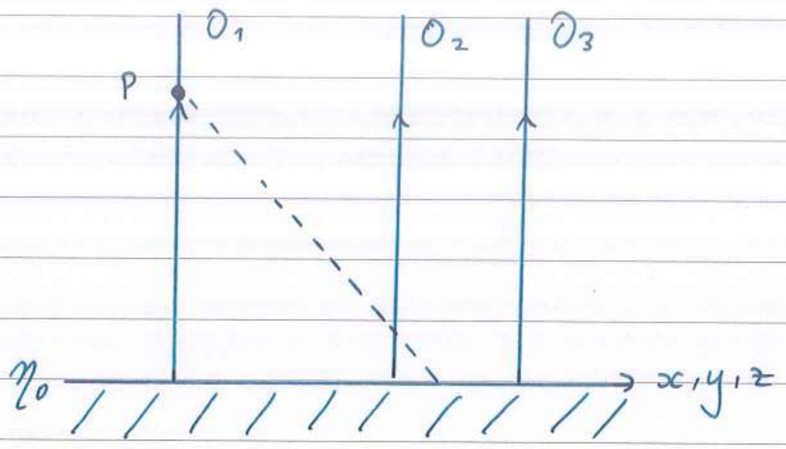
Otherwise there are <u>horizons</u>.

- If $\int_0 \frac{d\tau}{a}$ converges / is finite, there is a <u>past horizon</u> ~~(a grows as $\tau \to 0$, universe contracts going forward)~~

e.g. $\int_0^{\tau} \frac{d\tau}{a} = \eta(\tau) - \eta_0$

a "particle horizon"





O_1 at P can receive signal from O_2 but not O_3.

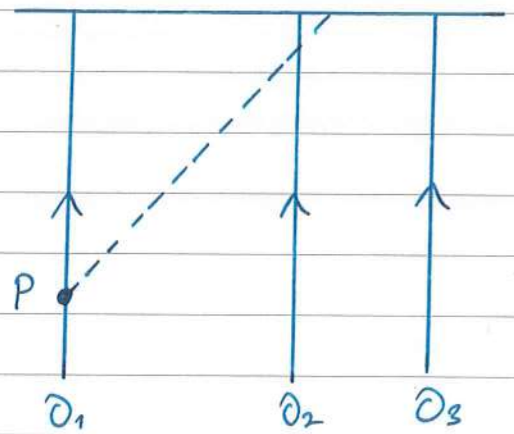
<u>Past horizon distance</u>:

$a(\tau_0)\int_0^{\tau_0} \frac{d\tau}{a(\tau)}$ is maximum physical distance at time τ_0 between observers in causal contact.

"distance travelled by light between τ_0 and start of universe"

- If $\int^{\infty} \frac{d\tau}{a}$ converges, there is a

<u>future horizon</u> (~~$a \to 0$ as $\tau \to \infty$~~



Observer at P can send signal to O_2 but can never communicate with O_3

Future horizon distance:

$$a(\tau_0) \int_{\tau_0}^{\infty} \frac{d\tau}{a}$$

is maximum physical distance at time τ_0 between observers that can communicate in the future.

Existence of the horizons depends on the form of $a(\tau)$ (so the matter content and $k = \pm 1, 0$).

Example: Dust $(p = 0)$

$$a(\tau) = \alpha \tau^{2/3}$$

- $$\int_0^{\tau_0} \frac{d\tau}{a} \sim \alpha \tau^{1/3} \Big|_0^{\tau_0}$$
  $$\sim \alpha \tau_0^{1/3}$$

  Converges so there is a past horizon

- $$\int_{\tau_0}^{\infty} \frac{d\tau}{a} \sim \alpha \tau^{1/3} \Big|_{\tau_0}^{\infty} \sim \infty - \alpha \tau_0^{1/3}$$

Diverges, no future horizon.

- Past horizon distance

$$a(\tau_0)\int_0^{\tau_0} \frac{d\tau}{a(\tau)} = \alpha \tau_0^{2/3} \int_0^{\tau_0} \frac{1}{\alpha} \tau^{-2/3} d\tau$$

$$= 3\tau_0$$

$$(= 3c\tau_0)$$

Defines the size of the observable universe.

<u>Example</u>: Cosmological constant, $\Lambda > 0$

$$a(\tau) = e^{\alpha\tau} \qquad \alpha := \sqrt{\frac{\Lambda}{3}}$$

Both integrals converge so future and past horizons

Past: $$a(\tau_0)\int_0^{\tau_0} \frac{d\tau}{a(\tau)} = \frac{1}{\alpha}\left(e^{\alpha\tau_0} - 1\right)$$

$$\simeq \frac{1}{\alpha} e^{\alpha\tau_0} \qquad \left(\tau_0 \gg \frac{1}{\alpha}\right)$$

Grows exponentially with τ_0 (can observe more of universe)

Future: $$a(\tau_0)\int_{\tau_0}^{\infty} \frac{d\tau}{a(\tau)} = \frac{1}{\alpha} e^{\alpha\tau_0}\left(e^{-\alpha\tau_0} - 0\right)$$

$$= \frac{1}{\alpha}$$

Constant! Cannot communicate with different distances as τ_0 changes