C7.5: General Relativity I
Problem Sheet 2

Please submit solutions in the collection boxes in the basement of the Mathematical Institute no later than Monday 5th November, 4pm. Corrections to `ashmore@maths.ox.ac.uk`.

1. Energy-momentum tensor of a perfect fluid

Consider some distribution of matter with energy momentum tensor T^{ab}. What is the 4-momentum per unit volume and directional pressure measured by an observer with 4-velocity V^a?

The energy-momentum tensor of a perfect fluid is given by

$$T^{ab} = (\rho + P)\, U^a U^b + P\eta^{ab},$$

where η^{ab} is the inverse metric in Minkowski space and U^a is the 4-velocity of the fluid. By considering an observer at rest with respect to the motion of the fluid, explain the physical meaning of ρ and P.

The equation of motion of a perfect fluid in an inertial frame is

$$\partial_a T^{ab} = 0.$$

In the remainder of the question you will show how the equations of fluid mechanics are compactly encoded in this single expression.

First show that the tensor

$$h^a{}_b = \delta^a{}_b + U^a U_b,$$

obeys

1. $h^a{}_b U^b = 0$
2. $h^a{}_b h^b{}_c = h^a{}_c$
3. $h^a{}_a = 3$

Using these properties, explain why $h^a{}_b$ is a projector onto the 3-dimensional hypersurfaces perpendicular to the fluid's 4-velocity U^a. What is the meaning of the symmetric tensor $h_{ab} = \eta_{ac} h^c{}_b$?

By projecting the equation of motion parallel and perpendicular to the 4-velocity of the fluid U^a, derive the equations

$$\partial_a(\rho U^a) + P\partial_a U^a = 0, \qquad (\rho + P)\frac{\mathrm{d}U^a}{\mathrm{d}\tau} + h^{ab}\partial_b P = 0, \tag{1}$$

where τ is the proper time of a particle moving with the fluid. The equations are the relativistic versions of the continuity and Euler equations of fluid mechanics. Show that the fluid particles move along geodesics when $P = 0$.

We now consider the non-relativistic approximation to equations (1). In the non-relativistic approximation you will need to assume that

1. $U^a = (1, \vec{u}) \qquad |\vec{u}| \ll 1$
2. $P \ll \rho$
3. $|\vec{u}|\,\partial_t P \ll |\vec{\nabla} P|$

What is the physical intuition behind each of the approximations? (It may be helpful to restore the speed of light c in the equations.) Using the approximation, show that

$$\partial_t \rho + \vec{\nabla}\cdot(\rho\vec{u}) = 0, \qquad \rho\big(\partial_t + \vec{u}\cdot\vec{\nabla}\vec{u}\big) = -\vec{\nabla}P.$$

2. Uniform acceleration and the equivalence principle

Let us start from a global inertial frame $\mathcal{O}$ in Minkowski space with coordinates $x^a = (t, x, y, z)$. Now consider the transformation to a *non-inertial* frame $\mathcal{O}'$ with coordinates $x'^a = (t', x', y', z')$ such that

$$\begin{aligned} t &= \left(\frac{1}{g} + z'\right)\sinh\left(gt'\right), \\ z &= \left(\frac{1}{g} + z'\right)\cosh\left(gt'\right) - \frac{1}{g}, \\ x &= x', \\ y &= y', \end{aligned}$$

where g is a constant with units of acceleration.

1. For $t' \ll 1/g$ show that this transformation corresponds to a uniformly accelerated reference frame in Newtonian mechanics.
2. Plot the trajectory of the point $z' = 0$ in the inertial frame $\mathcal{O}$.
3. Show that a clock at rest at $z' = h$ runs fast compared to a clock at rest at $z' = 0$ by the factor $(1 + gh)$.
4. Use the equivalence principle to interpret this result in terms of gravitational time dilation.
5. What is the line element $\mathrm{d}s^2$ for a spacetime with a uniform gravitational field?

3. Flat space in polar coordinates

Flat $\mathbb{R}^2$ has coordinates x^1 and x^2 and a metric with components $g_{11} = g_{22} = 1$, $g_{12} = g_{21} = 0$. Changing to polar coordinates defined by

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix},$$

1. Find the components of the metric.
2. Find the Christoffel symbols using:
   (a) The geodesic equation as derived from the point-particle Lagrangian.
   (b) The expression of the Christoffel symbols in terms of the metric
   $$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}).$$
   (c) The transformation behavior of the Christoffel symbols under a change of coordinates.
3. Show that straight lines in $\mathbb{R}^2$ solve the geodesic equation in polar coordinates.

4. The covariant derivative

Together with the usual properties of a derivative, the covariant derivative is defined as a map from (p, q) tensors to $(p, q+1)$ tensors. The action of the covariant derivative on a vector (a $(1, 0)$ tensor) with components v^a can be written as

$$\nabla_b v^a = \partial_b v^a + \Gamma^a{}_{bc} v^c.$$

1. Prove that $\nabla_b v^a$ transforms as a $(1, 1)$ tensor under general coordinate transformations provided the Christoffel symbols transform as
$$\Gamma'^a{}_{bc} = \frac{\partial x^p}{\partial x'^b} \frac{\partial x^q}{\partial x'^c} \left(\frac{\partial x'^a}{\partial x^r} \Gamma^r{}_{pq} - \frac{\partial^2 x'^a}{\partial x^p \partial x^q} \right).$$
2. Show that the above transformation behavior is implied by
$$\Gamma^a{}_{bc} = \tfrac{1}{2} g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}).$$
3. What is the action of the covariant derivative ∇_a on a scalar? Use this to show how ∇_a acts on a one-form ω_a.
4. What is the action of the covariant derivative ∇_a on a (p, q) tensor?

5. 2d de Sitter space

Consider the two-dimensional de Sitter metric

$$\mathrm{d}s^2 = -\mathrm{d}u^2 + \cosh^2 u \, \mathrm{d}\varphi^2,$$

where $-\infty < u < \infty$ and $0 \le \varphi < 2\pi$.

1. What is the proper length of the space-like curve defined by $u = u_\mathrm{c}$, where u_c is constant?

2. Write down the Lagrangian for (affinely parametrised) geodesics and, using Lagrange's equations, compute the non-vanishing Christoffel symbols:

$$\Gamma^u_{\varphi\varphi}, \qquad \Gamma^\varphi_{u\varphi}, \qquad \Gamma^\varphi_{\varphi u}.$$

3. Show that

$$J = \cosh^2 u \, \dot{\varphi}, \qquad E = \dot{u}^2 - \cosh^2 u \, \dot{\varphi}^2,$$

are both conserved along geodesics. *Hint: for an elegant derivation of the second conserved quantity, you may want to compute the Hamiltonian and explain why it is conserved.*

4. Consider a geodesic with initial condition $\dot{\varphi}(0) = 0$. Show that $J = 0$ and $E = \dot{u}^2$.

5. Now consider the case $J \neq 0$. Introduce the variable $v = \tanh u$ and show that

$$\left(\frac{\mathrm{d}v}{\mathrm{d}\varphi}\right)^2 = \frac{E}{J^2} + 1 - v^2.$$

Write down the most general solution $v(\varphi)$ and discuss its behaviour as a function of the ratio E/J^2.

6. Infinitesimal symmetries

The Lie derivative of a type $(2,0)$ tensor T_{ab} with respect to a vector field X^a is defined by

$$\mathcal{L}_X T_{ab} = X^c \partial_c T_{ab} + (\partial_a X^c) T_{cb} + (\partial_b X^c) T_{ac}.$$

1. Show that you can replace ∂_a with the Levi-Civita covariant derivative ∇_a in this expression, so that $\mathcal{L}_X T_{ab} = \mathcal{L}^\nabla_X T_{ab}$. Using this argue that the Lie derivative transforms as a tensor. What is $\mathcal{L}^\nabla_X Y^a - \mathcal{L}_X Y^a$ if Y^a is a $(1,0)$ tensor and ∇ is a general connection (do not assume Γ is symmetric in its lower indices)?

2. Consider the infinitesimal transformation $\delta x^a = \epsilon \, K^a$ generated by a vector field K^a. Show that the action of an (affinely parametrised) geodesic

$$S = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \, g_{ab} \frac{\mathrm{d}x^a}{\mathrm{d}\lambda} \frac{\mathrm{d}x^b}{\mathrm{d}\lambda},$$

is invariant under this transformation if

$$\mathcal{L}_K g_{ab} = 0.$$

Show that this can be equivalently expressed as

$$\nabla_{(a} K_{b)} = 0,$$

where ∇_a is the covariant derivative associated to the metric g_{ab} (the Levi-Civita connection for g). A vector field with this property is known as a *Killing vector* and generates an infinitesimal symmetry of the geometry defined by the metric g_{ab}.

3. Show that the inner product $g_{ab}K^a\dot{x}^b$ is conserved along the geodesic.

4. Consider the two-dimensional de Sitter metric from problem 4. Show that the vector field K^a with components $K^u = 0$ and $K^\varphi = 1$ is a Killing vector and the corresponding conserved quantity is J.

5. Show that the sum $X^a + Y^a$ and commutator $[X, Y]^a = X^b\nabla_b Y^a - Y^b\nabla_b X^a$ of two Killing vectors are also Killing vectors. Together with the Jacobi identity, this means that Killing vectors generate a Lie algebra.

7. Maxwell equations in general coordinates

In a general spacetime, the sourceless Maxwell equations are given by

$$\nabla_a F^{ab} = 0, \qquad \nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} = 0,$$

where $F_{ab} = -F_{ba}$. The Maxwell energy-momentum tensor is as

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F^c{}_b + \frac{1}{4}F^{cd}F_{cd}\, g_{ab}\right).$$

1. Show that

$$\nabla^a T_{ab} = 0.$$

2. Show that $\Gamma^b{}_{ab} = \partial_a \log\sqrt{-|g|}$, where $|g|$ is the determinant of the metric.

3. Show that the first Maxwell equation can be written as

$$\partial_a\left(\sqrt{-|g|}F^{ab}\right) = 0.$$

4. Show that we can substitute partial derivatives for covariant derivatives in the second Maxwell equation.

8. (Optional) Action principles and Lagrangians

This question is optional and will not be marked. It explores the relation between the two action principles we have discussed in the lectures.

The action for a massive free particle is given by

$$S_0[x] = -m \int \mathrm{d}\tau = -m \int \mathrm{d}\lambda \left(-g_{ab} \frac{\mathrm{d}x^a}{\mathrm{d}\lambda} \frac{\mathrm{d}x^b}{\mathrm{d}\lambda} \right)^{1/2},$$

where m is the particle rest mass and τ is proper time. This action cannot describe massless particles, instead we can use

$$S_1[x] = \int \mathrm{d}\lambda \, \mathcal{L} = \int \mathrm{d}\lambda \, g_{ab} \frac{\mathrm{d}x^a}{\mathrm{d}\lambda} \frac{\mathrm{d}x^b}{\mathrm{d}\lambda}.$$

Both of these actions give rise to the same equations of motion, but S_1 can also be used to describe massless particles.

1. Convince yourself that S_0 is invariant under reparametrisations $\lambda \mapsto \lambda'(\lambda)$ while S_1 is invariant under only $\lambda \mapsto \lambda + a$, where a is constant.

Consider a "parent action" S

$$S[x, e] = \frac{1}{2} \int \mathrm{d}\lambda \left(e^{-1} g_{ab} \frac{\mathrm{d}x^a}{\mathrm{d}\lambda} \frac{\mathrm{d}x^b}{\mathrm{d}\lambda} - m^2 e \right),$$

where we have introduced an additional field $e = e(\lambda)$.

2. Assuming e transforms under reparametrisations such that $e(\lambda)\mathrm{d}\lambda$ is invariant (a worldline one-form), convince yourself that S is invariant under $\lambda \mapsto \lambda'(\lambda)$.

We will now see how S reduces to S_0 or S_1.

3. As e appears algebraically in the action (without derivatives), you can find its equation of motion and substitute it back into the action. Convince yourself that the equation of motion is
$$g_{ab} \frac{\mathrm{d}x^a}{\mathrm{d}\lambda} \frac{\mathrm{d}x^b}{\mathrm{d}\lambda} + m^2 e^2 = 0,$$
and so you can solve for e providing $m \neq 0$. Using this, show that S reduces to S_0 and so argue that S_0 is valid only for massive particles. Convince yourself that as we did not fix the reparametrisation invariance, S_0 must have the same amount of reparametrisation invariance as S.

4. Convince yourself that as $e(\lambda)\mathrm{d}\lambda$ is invariant under reparametrisations, you can always pick a parametrisation or "gauge" in which $e(\lambda) = 1$. In this gauge, show that S reduces to S_1 up to a constant shift. Convince yourself that as we used some of the reparametrisation freedom to reach S_1, the resulting action will have a smaller symmetry ($\lambda \mapsto \lambda + a$ in this case).