C7.5: General Relativity I
Problem Sheet 4

Solutions will be posted online during 0^{th} week of Hilary term. Please email your tutors before the final class with any questions you would like to go through or other questions that come up in the course of your revision. Questions with a * are designed to be more challenging and may require extra reading.

1. Kepler's 3rd law

What is the physical meaning of the coordinate time t in the Schwarzschild solution?

Show that the proper time τ of an observer on a circular orbit at radius R and the coordinate time t are related by

$$\left(\frac{dt}{d\tau}\right)^2 = \frac{1}{1-3M/R}.$$

Using this result, show that

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = \frac{M}{R^3},$$

for a circular orbit at radius R.

2. Stability of circular orbits

Consider a time-like geodesic in the Schwarzschild geometry that is a small perturbation from a circular orbit at radius R in the equatorial plane $\theta = \pi/2$

$$r(\tau) = R + \varepsilon(\tau).$$

Show that the perturbation must solve an equation of the form

$$\ddot{\varepsilon} + f(R)\varepsilon = 0,$$

and find the function $f(R)$. Plot this function, showing clearly its asymptote and intercept. Hence, re-derive the fact that circular orbits only exist if $R > 3M$ and show that these orbits are stable if $R > 6M$.

3. Meaning of $\boldsymbol{E}$

Consider a stationary observer at radius R in the Schwarzschild geometry and a massive test particle moving on a time-like geodesic $x^a(\tau)$ that intersect at some point P. Show that the stationary observer measures the energy per-unit-rest mass of the test particle to be

$$\sqrt{1-\frac{2M}{R}}\,\dot{t}.$$

Let the test particle and the stationary observer have relative velocity v at the point P. Explain why

$$\gamma(v)=\sqrt{1-\frac{2M}{R}}\,\dot{t}.$$

Now derive an expression for the conserved quantity E. Expanding this expression for large distances ($R \gg 2M$) and small velocities ($v \ll 1$), show that E is approximately the sum of the rest mass, kinetic energy and potential energy.

This is an approximate characterization of the conserved quantity E. To find the precise meaning, suppose the stationary observer converts the energy measured into a photon and sends it out to another stationary observer at $r \to \infty$. What is the energy of the photon measured by a stationary observer at infinity?

4. Acceleration

Observers not freely falling experience acceleration forces. This is encoded in the acceleration 4-vector $A^a = V^b \nabla_b V^a$, where V^a is the 4-velocity. This measures the failure of the corresponding curve to be a geodesic.

Consider a stationary observer at radius R in the Schwarzschild geometry and show that their acceleration four-vector is

$$A^a = (0, m/R^2, 0, 0).$$

You will need to compute the Christoffel symbol $\Gamma^r{}_{tt}$ for the Schwarzschild metric from the r equation of motion. Now compute the proper acceleration $a = (g_{ab}A^a A^b)^{1/2}$. Show that this agrees with the Newtonian expectation for $r \gg 2M$ and that stationary observers can only exist for $R > 2M$. Which of these results is more physical?

5. Capture by a black hole

For geodesics in the Schwarzschild solution

$$\frac{E^2-\kappa}{2}=\frac{1}{2}\dot{r}^2+V(r),$$

where

$$V(r)=-\frac{\kappa M}{r}+\frac{J^2}{2r^2}-\frac{MJ^2}{r^3},$$

with $\kappa = 0$ for null geodesics and $\kappa = 1$ for time-like geodesics

In this question we are interested in when incoming geodesics will be captured by a black hole. For such problems it is convenient to define the *impact parameter* b by

$$b := \frac{J}{\sqrt{E^2-\kappa}}.$$

a) Massless particle

First consider an incoming null geodesic. Show that a massless particle is captured by the black hole if the impact parameter b is smaller than a critical value b_c. Show that the *capture cross-section* $\sigma := \pi b_\mathrm{c}^2$ is

$$\sigma = 27\pi M^2.$$

b) Non-relativistic massive particle

Now consider an incoming time-like geodesic. We will assume that the massive particle starts at $r = \infty$ with non-relativistic velocity $v \ll 1$ measured by a stationary observer. Explain why

$$b = \frac{J}{v} + \mathcal{O}(v),$$

and draw a diagram explaining the physical significance of the impact parameter in this case.

Show that the massive particle will be captured by the black hole if the impact parameter b is smaller than a critical value b_c. Show that the capture cross-section $\sigma := \pi b_\mathrm{c}^2$ is approximately

$$\sigma = 16\pi M^2/v^2.$$

6. Einstein equations in cosmology

The FRW metric in coordinates (τ, r, θ, ϕ) is

$$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{d}\tau^2 + a(\tau)^2 \mathrm{d}\Sigma_{(3)}^2, \\ \mathrm{d}\Sigma_{(3)}^2 &= \frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\Omega^2 \end{aligned}$$

Using this metric, show that the Einstein equations in the presence of a perfect fluid imply

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2}, \qquad \frac{a''}{a} = -\frac{4\pi}{3}(\rho + 3P).$$

Multiply the first equation by a^2, differentiate with respect to τ and eliminate a'' using the second equation to show that

$$\rho' + 3\frac{a'}{a}(\rho + P) = 0.$$

Derive the same equation directly from the local conservation of energy and momentum $\nabla^a T_{ab} = 0$.

7. Cosmological constant

A cosmological constant Λ modifies Einstein's equations as follows

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi T_{ab}.$$

Show that the cosmological constant is mathematically equivalent to a perfect fluid with density $\rho_\Lambda = \Lambda/8\pi$ and pressure $P_\Lambda = -\Lambda/8\pi$. Hence show that for cosmological solutions we have

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}, \qquad \frac{a''}{a} = -\frac{4\pi}{3}(\rho + 3P) + \frac{\Lambda}{3}.$$

In an expanding universe with contributions from pressureless matter, radiation and a cosmological constant, which contribution will dominate the energy density at a) early times; b) late times? Consider a universe with a positive cosmological constant: how does the scalar factor $a(\tau)$ behave at late times? Does this universe have a future horizon?

8.* Conformal transformations

A conformal transformation of a spacetime is one where the metric g_{ab} of the original spacetime is transformed into the metric $\tilde{g}_{ab}$ of a new spacetime, such that they are related by

$$\tilde{g}_{ab} = \Omega^2 g_{ab},$$

where Ω is a function of the spacetime coordinates x^a. Compute $\tilde{\Gamma}^a{}_{bc}$ and show that $\tilde{\Gamma}^a{}_{bc} = \Gamma^a{}_{bc}$ if and only if Ω is constant.

Suppose in the original spacetime one has a solution to the source-free Maxwell equations

$$\nabla_a F^{ab} = 0, \qquad \nabla_{[a} F_{bc]} = 0.$$

Show that F_{ab} is also a solution to the source-free Maxwell equations in the new spacetime with metric $\tilde{g}_{ab}$ (with a corresponding connection $\tilde{\nabla}_a$). You may wish to use $\Gamma^a{}_{ab} = (-g)^{-1/2}\partial_b(-g)^{1/2}$ where $g = \det g_{ab}$.

The metric for a flat FRW spacetime ($k = 0$) is sometimes written as

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \left(\frac{\tau}{\tau_0}\right)^{2/3}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2).$$

Show that this spacetime is conformal to Minkowski spacetime.

Using this, or otherwise, find a formula for the cosmological red shift of light emitted by a galaxy at $\tau = \tau_1$ and measured by an observer at $\tau = \tau_2$. Assume both the source and observer are comoving.

9.* Energy conditions

The *weak energy condition* (WEC) requires that any stress-energy tensor must satisfy

$$T_{ab}t^a t^b \geq 0,$$

for all timelike vectors t^a. Show that for a perfect fluid the WEC implies

$$\rho \geq 0, \qquad p + P \geq 0.$$

What type of matter would violate the WEC?

The *strong energy condition* (SEC) requires that any stress-energy tensor must satisfy

$$T_{ab}t^a t^b \geq \tfrac{1}{2}T^a{}_a t^b t_b,$$

for all timelike vectors t^a. Show that for a perfect fluid the SEC implies

$$\rho + P \geq 0, \qquad \rho + 3P \geq 0.$$

Does the SEC imply the WEC? Is the SEC satisfied for a flat FRW universe with a positive cosmological constant?

Starting from the Einstein equations, show that the SEC implies

$$R_{ab}t^a t^b \geq 0,$$

where R_{ab} is the Ricci tensor.

10.* Cosmic strings

Consider a static, infinitely long, cylindrically symmetric matter distribution of constant radius that is invariant under Lorentz boosts along the symmetry axis. Using similar arguments that lead to the Schwarzschild solution, show that the metric outside the matter distribution can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + (\alpha + \beta r)^2 \mathrm{d}\phi^2 + \mathrm{d}z^2,$$

where r is the radial direction, z is the direction along the symmetry axis, and α and β are constants.

For the case $\alpha = 0$, consider the spacelike surfaces defined by $t =$ const.and $z =$ const. Calculate the circumference of a circle of constant coordinate radius r in such a surface.

Argue that the for $\beta < 1$, the geometry on the spacelike surface is that of a two-dimensional cone embedded in three-dimensional Euclidean space.

10.* Totally antisymmetric torsion

Let ∇_a be a torsion-free connection and define a new connection $\tilde{\nabla}_a$ such that

$$\tilde{\nabla}_a V^b = \nabla_a V^b - \tfrac{1}{2}T^b{}_{ac}V^c,$$

for any vector field V^a, where $T^b{}_{ac} = -T^b{}_{ca}$. Let $\Omega_{ab} = -\Omega_{ba}$ be a non-degenerate antisymmetric tensor with inverse $\hat{\Omega}^{ac}$ such that

$$\Omega_{ab}\hat{\Omega}^{bc} = \delta^c_a.$$

Show that there exists a unique choice for $T^b{}_{ac}$ such that

$$\tilde{\nabla}_a \Omega_{bc} = 0.$$