Calabi-Yau Metrics, CFTs and Random Matrices

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Overview

Calabi–Yau metrics are important for both string phenomenology and CFTs

The Laplacian encodes both geometry and the spectrum of operators in certain 2d CFTs

Numerical methods give us access to this data

For CY CFTs, averaging over complex structure, this spectrum displays random matrix statistics

Motivation

Does string theory describe our universe?

- Heterotic string on Calabi–Yau comes closest to realistic MSSM models

Usually focus on getting correct gauge group, matter spectrum, superpotential, etc.

- Do not need details of metric for these

How many of these string vacua are physically reasonable?

String theory

Problem: a 'theory of everything' should give particle masses, couplings, supersymmetry breaking patterns, etc.

- Need Kähler potential of theory and zero modes – eigenmodes of Laplacian Δ depend on the metric on the internal space
- No explicitly known (compact) Calabi–Yau metrics

This talk: computing these metrics numerically and applications to CFTs

Calabi–Yau compactifications

Minimal SUSY on $\mathbb{R}^{1,3} \times X$ with gauge bundle V [Candelas et al. '85]

- No H flux $\Rightarrow$ X is Calabi–Yau, V admits HYM connection

Physics in 4d determined by geometry of X – Kaluza–Klein reduction fixes 4d modes

- e.g. for KK scalars, masses in 4d c.f. eigenvalues of Laplacian in 6d

$$\Delta\phi_6 = \lambda\phi_6 \quad \Rightarrow \quad \Box_4\zeta_4 = \lambda\zeta_4 \equiv m^2\zeta_4$$

- Zero modes determine low-energy physics, e.g. matter fields c.f. harmonic representatives of $H^1(X, V_R)$

Physics from geometry

Particle content comes from choice of X and V

- $\mathrm{SU}(3)$ bundle gives E_6 GUT gauge group in 4d with $\frac{1}{2}\chi(X)$ particle generations

Most interesting examples c.f. non-standard embedding which give MSSM

Need superpotential and Kähler potential of 4d theory, e.g. matter fields in $H^1(X, V_{\mathbf{R}})$ with harmonic basis $\{\psi_a\}$

- Superpotential and Kahler potential from overlap of zero modes on X

$$\lambda_{abc} = \int_X \Omega \wedge \mathrm{tr}(\psi_a \wedge \psi_b \wedge \psi_c), \qquad G_{a\bar{b}} \sim \int_X \psi_a \wedge \star_V \bar{\psi}_{\bar{b}}$$

Numerical CY metrics and spectra

Calabi–Yau basics

Calabi–Yau manifolds are Kähler manifolds with a Ricci-flat metric

- Existence but no explicit constructions – $c_1(X) = 0 \Rightarrow$ there exists a Ricci-flat metric [Yau '77]

Kähler $\Rightarrow$ Kähler potential K gives (real) closed two-form $J = \partial\bar{\partial}K$

$c_1(X) = 0 \Rightarrow$ (complex) nowhere-vanishing closed (3,0)-form Ω

$$J^3 = \text{vol}_J, \qquad |\Omega|^2 = \text{vol}_\Omega.$$

Example: Fermat quintic

Quintic hypersurface Q in $\mathbb{P}^4$

$$Q(z) \equiv z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0$$

(3,0)-form Ω determined by Q, e.g. in $z_0 = 1$ patch

$$\Omega = \frac{\mathrm{d}z_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}z_4}{\partial Q / \partial z_1}$$

Metric g and Kähler form J determined by Kähler potential

$$g_{i\bar{j}}(z,\bar{z}) = \partial_i \bar{\partial}_{\bar{j}} K(z,\bar{z}), \qquad \mathrm{vol}_J \sim \det g_{i\bar{j}} \mathrm{d}^6 z$$

How do we measure accuracy?

The Ricci-flat metric is given by a K that satisfies (c.f. Monge–Ampère)

$$\left.\frac{\mathrm{vol}_J}{\mathrm{vol}_\Omega}\right|_p = \text{constant} \quad \Rightarrow \quad R_{i\bar{j}} = 0$$

Define a functional of K

$$\sigma(K) = \int_X \left| 1 - \frac{\mathrm{vol}_J}{\mathrm{vol}_\Omega} \right| \mathrm{vol}_\Omega$$

The exact CY metric has $\sigma = 0$

Finding the Ricci-flat metric reduces to finding a single function $K(z, \bar{z})$ that minimises σ

Algebraic metrics

Natural Kähler metric on $\mathbb{P}^4$ given by

$$K_{\text{FS}} = \log \sum_{i=0}^{4} z_i \bar{z}_{\bar{i}}$$

Can generalise this with a hermitian matrix $h^{i\bar{j}}$

$$K(h) = \log \sum_{i,\bar{j}=0}^{4} z_i h^{i\bar{j}} \bar{z}_{\bar{j}}$$

Restricting to $Q \subset \mathbb{P}^4$ (defined by $Q = 0$) gives a Kähler metric but not Ricci-flat

- 25 real parameters in $h^{i\bar{j}}$ that we can vary
- Need more parameters to better approximate the Ricci-flat metric

Algebraic metrics

Generalisation of Fubini–Study: replace coordinates z_i with homogeneous polynomials s_α of degree k

$$\text{e.g.} \quad k = 2: \quad s_\alpha = (z_0^2, z_0 z_1, z_0 z_2, \ldots)$$

Kähler potential is then

$$K(h) = \log \sum_{\alpha,\bar\beta=0}^{14} s_\alpha h^{\alpha\bar\beta} \bar{s}_{\bar\beta}, \qquad h^{\alpha\bar\beta} \sim 225 \text{ parameters}$$

At degree k have $\mathcal{O}(k^4)$ parameters, so can approximate the Ricci-flat metric to arbitrary precision

- Algebraic metrics [Tian '90] – higher k allows better precision
- Spectral method as $s_\alpha \bar{s}_{\bar\beta}$ give a basis for eigenspaces on $\mathbb{P}^4$

How to fix $h^{\alpha\bar{\beta}}$?

Finding the “best” approximation to the Ricci-flat metric amounts to finding $h^{\alpha\bar{\beta}}$ so that σ is minimised

Three approaches:

- Iterative procedure [Donaldson ‘05; Douglas ‘06; Braun ‘07]
- Minimise σ directly [Headrick, Nassar ‘09]
- Treat σ as a loss function for a neural network [Anderson et al. ‘20]

One can also try to find K or $g_{i\bar{j}}$ directly [Headrick, Wiseman ‘05; Douglas et al. 20; Anderson et al. ‘20; Jejjala ‘20]

In all cases, numerical integrals carried out by Monte Carlo

The Laplacian [Braun et al. '08, AA '20]

Operators in CFT determined by eigenmodes on CY

Eigenmodes are (p, q)-eigenforms of the Laplacian

$$\Delta = \mathrm{d}\delta + \delta\mathrm{d}, \qquad \Delta|\phi_n\rangle = \lambda_n|\phi_n\rangle$$

where λ_n are real and non-negative and can appear with multiplicity (c.f. continuous or finite symmetries)

- Need some way of computing the spectrum (and the harmonic modes themselves for pheno)

The Laplacian

Given a (non-orthonormal) basis of functions $\{\alpha_A\}$, we can expand the eigenmodes as

$$|\phi\rangle = \sum_A \langle\alpha_A|\tilde{\phi}\rangle \, |\alpha_A\rangle, \qquad A = 1, \ldots, \dim\{\alpha_A\}$$

so that $\Delta|\phi\rangle = \lambda|\phi\rangle$ becomes generalised eigenvalue problem for λ and $\tilde{\phi}_A$

$$\begin{aligned} \langle\alpha_A|\Delta|\alpha_B\rangle\langle\alpha_B|\tilde{\phi}\rangle &= \lambda\langle\alpha_A|\alpha_B\rangle\langle\alpha_B|\tilde{\phi}\rangle \\ \Rightarrow \qquad \Delta_{AB}\tilde{\phi}_B &= \lambda O_{AB}\tilde{\phi}_B \end{aligned}$$

where

$$O_{AB} = \int \alpha_A \wedge \star\bar{\alpha}_B, \quad \text{etc.}$$

The Laplacian

Basis $\{\alpha_A\}$ is infinite dimensional – truncate to a finite approximate basis at degree k_ϕ in z_i

$$\{\alpha_A\} = \frac{(\text{degree } k_\phi \text{ function})\overline{(\text{degree } k_\phi \text{ function})}}{(|z_0|^2 + \ldots |z_4|^2)^{k_\phi}}$$

(c.f. harmonic functions on $\mathbb{P}^4$)

1. Compute matrices Δ_{AB} and O_{AB} numerically for independent choices of (p, q)
2. Find eigenvalues and eigenvectors

Toy example: $d = 2$

Two-dimensional flat tori are Calabi–Yau and their spectrum can be computed *explicitly* [Milnor '63]

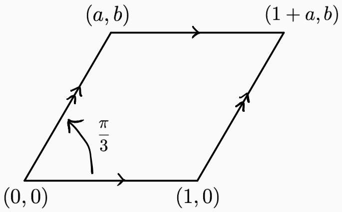
- Defined by $\tau \equiv a + ib$ where lattice generated by $(1, 0)$ and (a, b)

Eigenvalues are

$$\lambda = 4\pi^2 b\left[\left(1 + \frac{a^2}{b^2}\right)m^2 - \frac{2a}{b^2}mn + \frac{n^2}{b^2}\right], \qquad m, n \in \mathbb{Z}$$

Toy example: $d = 2$

Consider the equilateral torus defined by $\tau = e^{i\pi/3}$ – $(1, 0)$ and (a, b) generate a hexagonal lattice



Equivalent to the curve in $\mathbb{P}^2$ defined by

$$Q \equiv z_0^3 + z_1^3 + z_2^3 = 0$$

Both have obvious $\mathbb{Z}_3$ symmetries

- Can now check numerics against *known* result

Toy example: $d = 2$

On an arbitrary CY we don't have the exact metric:

1. Specify the CY by $Q = 0$ and compute metric numerically
2. Pick a finite basis for (p, q)-forms at some degree k_ϕ
3. Solve numerically for eigenvalues and eigenmodes of Laplacian for each choice of (p, q) using Monte Carlo to evaluate integrals

Compute these using $k_\phi = 6$ for Laplacian

- $N = 10^6$ for both metric and Laplacian
- $k_\phi = 6 \Rightarrow \dim\{\alpha_{0,0}\} = 576$ and $\dim\{\alpha_{1,0}\} = \dim\{\alpha_{1,1}\} = 324$
- Metric in ~ 1 min, $(1, 0)$ in ~ 30 sec, $(0, 0)$ or $(1, 1)$ in ~ 1 min

Toy example: $d = 2$

n	λ_n			Exact	
	$(0,0)$	$(1,0)$	$(1,1)$	λ_n	μ_n
0	0.00	0.00	10^{-5}	0.00	1
1	45.6	45.6	45.6	45.6	6
2	136.7	136.7	136.7	136.8	6
3	182.2	182.3	182.3	182.3	6
4	318.9	319.0	319.0	319.1	12
⋮	⋮	⋮	⋮	⋮	⋮
20	2183	2186	2186	2188	6
21	2232	2235	2235	2234	18

N.B. $\lambda_1 = 8\pi^2/\sqrt{3} \approx 45.6$ is the maximum for any metric on T^2 [Berger '73; Nadirashvili '96]

We see that

$$\lambda^{(1,0)} = \lambda^{(0,0)} = \lambda^{(1,1)}$$

Why?

- Hodge star commutes with Δ and so $\lambda^{(0,0)} = \lambda^{(1,1)}$
- ∂ commutes with Δ so Hodge decomposition $\phi_{1,0} = \gamma_{1,0} + \partial\phi_{0,0}$ plus $h^{1,0} = 1$, implies $\lambda^{(1,0)} = \lambda^{(0,0)}$

CY CFTs and RMT

2d conformal field theories

Spectrum of conformal field theories

- Most interacting CFTs understood near special points in moduli space, e.g. K3 as $T^4/\mathbb{Z}_2$
- Most information is about quantities protected by supersymmetry, e.g. counts of BPS objects, or using modular invariance [Witten '82; ...,Keller, Ooguri '12; ...]

CYs appear as target spaces for CFTs: spectrum of operators encoded in geometry

- In large-volume limit, low-lying modes c.f. quantum mechanics with $H = \Delta$ [Witten '82]

Chaos in 2d CFTs?

Spectrum of a 2d CFT quantised on sphere determined by

$$H|\mathcal{O}_i\rangle = D_i|\mathcal{O}_i\rangle, \qquad D_i \geq 0$$

Question

Given an ensemble of CFTs, what are the statistics of the scaling dimensions $\{D_i\}$?

Need spectrum of *generic interacting CFTs* (not solvable/rational/etc.) that come in families

- Not possible until recently! (though see [Afkhami-Jeddi et al. '06; Maloney, Witten'20; Benjamin et al. '21] for free theories)

σ-models and CFTs

Consider CFT defined by σ-model with Calabi–Yau target X (irrational, not solvable)

$$c = 3 \dim_{\mathbb{C}} X$$

Well-understood using mirror symmetry, supersymmetry, etc. – but now want non-BPS data!

These CFTs come in families labelled by

$$(\text{Kähler moduli}, \text{complex structure moduli})$$

Varying moduli $\rightarrow$ ensemble of CFTs

Large-volume limit

In large-volume limit, spectrum of operators

$$\mathcal{O} = \mathcal{O}_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} \lambda^{i_1} \dots \lambda^{i_p} \bar{\psi}^{\bar{j}_1} \dots \bar{\psi}^{\bar{j}_q}$$

corresponds to (p, q)-eigenforms of Δ for Calabi–Yau metric on X

Quantum numbers are

$$D = \Delta + \frac{p+q}{2}, \qquad J = \frac{p-q}{2}.$$

$\Delta \sim \mathrm{vol}^{-1/\dim_{\mathbb{C}} X}$ so at large volume, light operators come from scalar eigenmodes of Δ

Ensembles of CYs

Generic quintic threefold given by quintic equation in $\mathbb{P}^4$

$$Q \equiv \sum_{m,n,p,q,r} c_{mnpqr} z_m z_n z_p z_q z_r = 0$$

101 complex structure parameters

Choose the c_{mnpqr} randomly from disk in complex plane

$$c_{mnpqr} \in \mathbb{C}, \qquad |c_{mnpqr}| < 1$$

Then compute the approximate CY metric and the spectrum of the scalar Laplacian for each instance

Plan

1. Numerically compute the CY metric for some choice of moduli
2. Numerically compute the spectrum of Δ (lowest ~ 100 eigenvalues)
3. Repeat for different choice of complex structure moduli $\rightarrow$ ensemble of CFT data
4. Compare statistics of ensemble to random matrices

Random matrix theory (RMT)

Random matrix statistics are a hallmark of quantum chaos

BGS conjecture: systems with ergodic classical limits display RMT statistics in their quantum energy levels [Bohigas, Giannoni, Schmit '84]

- Spectrum exhibits level repulsion and long-range rigidity

RMT has appeared in nuclear physics, billiards, SYK model and black hole physics / quantum gravity

- BH energy levels are discrete, non-degenerate and chaotic
  [Maldacena '01; Cotler et al. '16; Saad et al. '18; ...]

Holography suggests that generic CFTs might display chaos

Spectral statistics

Want to compare eigenvalue statistics with universal features of random matrix theory

Gaussian orthogonal ensemble (GOE) $=$ $N \times N$ real symmetric matrices

e.g. density of eigenvalues (large N) given by Wigner's semicircle

$$\rho(\lambda) = \frac{1}{\pi}\sqrt{2N - \lambda^2}$$

This is not a universal feature of chaotic system. Instead interested in statistics after normalising $\rho = 1$

- Unfolded spectrum focuses on fluctuations

Level repulsion

RMT displays eigenvalue repulsion – probability of distance s between consecutive eigenvalues is

$$p_1(s) = \frac{\pi}{2} s e^{-\frac{\pi}{4}s^2}$$

Nearest-neighbour level spacing – peaked away from origin $\Rightarrow$ eigenvalues repel!

- Poisson statistics $p_1(s) = e^{-s}$ if *not* chaotic
- $p_1(s)$ depends on all n-pt correlation functions

Spectral rigidity

Two-point function given by

$$G(s) = 1 - \frac{\sin^2(\pi s)}{\pi^2 s^2} - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\sin(\pi s)}{\pi s}\right)\int_s^\infty \mathrm{d}s' \, \frac{\sin(\pi s')}{\pi s'}$$

The connected correlator $G(s) - 1$ decays as s^{-2}

Spectral rigidity seen in fluctuation of the number of eigenvalues in a typical interval L

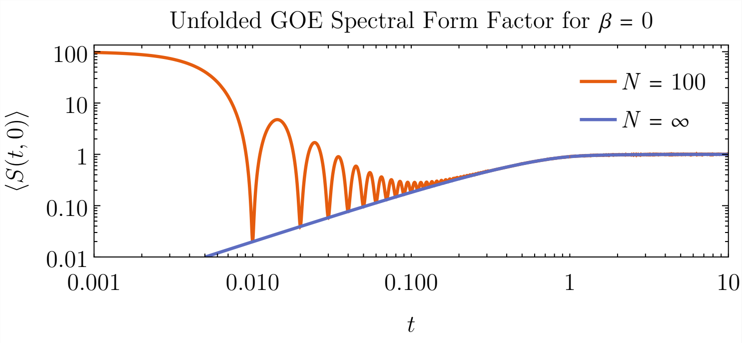
$$\Sigma^2(L) \sim \log L$$

where Poisson has linear growth $\Sigma^2(L) \sim L$

Spectral form factor

SFF is Fourier transform of two-point function

$$S(t,\beta) \sim \left|\sum_i e^{-(\beta+2\pi i t)\lambda_i}\right|^2 \sim \frac{1}{2\beta} + \frac{1}{\beta}\,\mathrm{Re}\int_0^\infty \mathrm{d}s\,(G(s)-1)e^{-(\beta+2\pi i t)s}$$



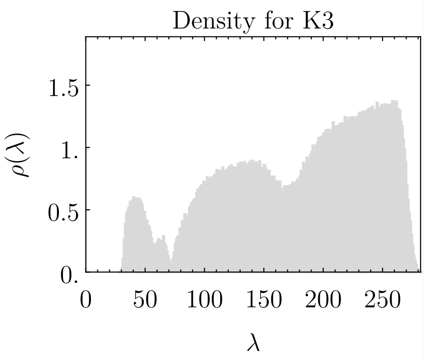
Dip → ramp → plateau

Results

Can then compare eigenvalue statistics for Calabi–Yau CFTs with RMT

- K3s as quartic equations in $\mathbb{P}^3$
- Quintic threefolds as quintic equations in $\mathbb{P}^4$

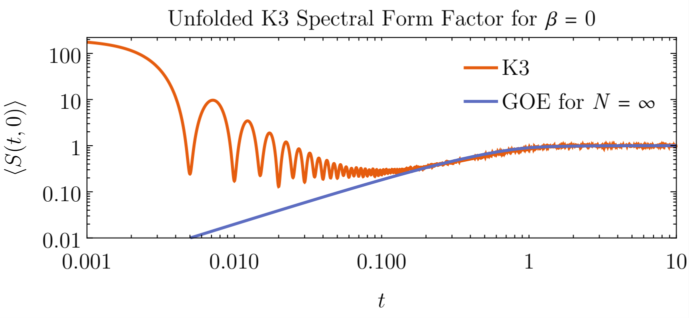
e.g. eigenvalue density for K3



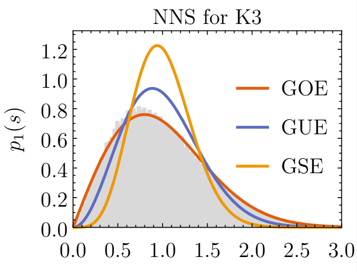
Not a semicircle! Fine, since that is not a *universal* feature

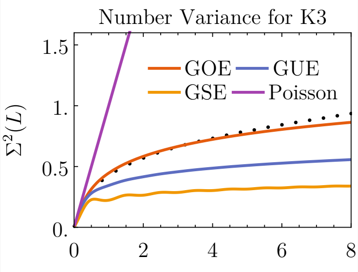
Instead should compare unfolded statistics

K3 statistics

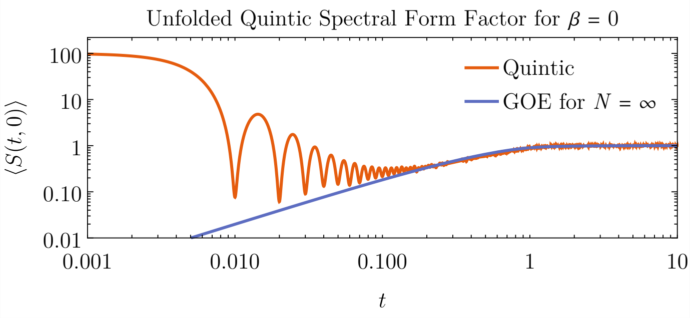


SFF shows dip, ramp and plateau expected from GOE





Quintic statistics



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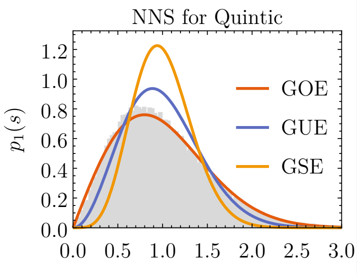

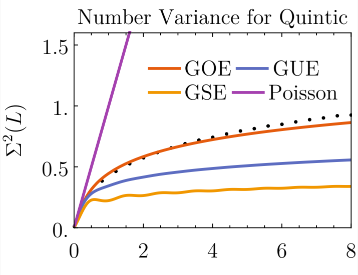

Summary and outlook

Calabi–Yau metrics are accessible with numerical methods

Spectrum is source of interesting non-BPS "data" with uses in geometry and CFTs

Spectrum of light operators in large-volume CFTs described by GOE statistics

- Other spectral statistics – spectral gap? eigenvalue density?
- Mirror symmetry in non-BPS spectrum? Modularity of 2d CFT?
- Typical compactifications? Distribution of Yukawa couplings? How many string vacua are physically acceptable?

Thank you!