# Marginal Deformations from Generalised Geometry


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# The idea

Generalised geometry is useful for understanding supersymmetric backgrounds

- Classic result in field theory on moduli spaces
- Marginal deformations of field theory gives deformations of AdS geometry and fluxes
- Can we realise field theory results in supergravity?

# Outline


1. Introduction
2. Generalised geometry
3. Generalised structures
4. Marginal deformations

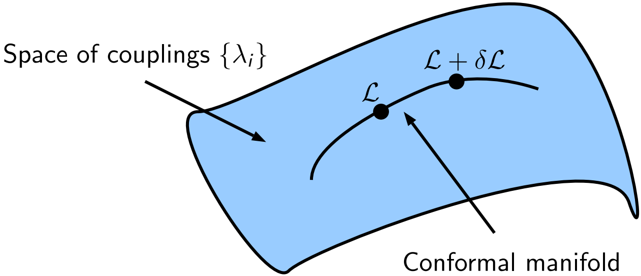
# Conformal field theories

CFTs are fixed points of RG flow

- Beta functions vanish
- Usually isolated points in space of couplings

SCFTs are special: admit conformal manifold of fixed points

- Deform by $\delta\mathcal{L} = \sum_i \lambda_i \mathcal{O}_i$



# SCFTs in four dimensions

Consider an $\mathcal{N} = 1$ SCFT in 4d. Classically, an operator $\mathcal{O}$ is marginal if

- Dimension $\Delta = 4$
- Coupling $\lambda$ is dimensionless

Couplings can run under RG flow and change dimension

- Beta function for coupling must vanish
- Operator is then exactly marginal

Exactly marginal couplings define conformal manifold

# Example: $\mathcal{N} = 4$ super Yang–Mills

Exactly marginal deformations preserve $\mathcal{N} = 1$

- Superpotential deformations from chiral superfields $\Phi_i$

$$\delta\mathcal{L} = f^{ijk}\,\mathrm{tr}(\Phi_i\Phi_j\Phi_k)$$

$f^{ijk}$ is complex symmetric tensor of SU(3) – ten marginal deformations

Compute one-loop beta functions

$$\gamma^i{}_j = f^{ikl}\bar{f}_{jkl} - \tfrac{1}{3}\delta^i{}_j f^{klm}\bar{f}_{klm} = 0$$

Leaves two exactly marginal deformations [Leigh, Strassler]

# General prescription

Why do the deformations have this form? [Green, Komargodski, Seiberg, Tachikawa, Wecht; Kol]

$$\delta\mathcal{L} = \sum_i \lambda_i \mathcal{O}^i$$

$\mathcal{N} = 1$ theory has $\mathrm{U}(1)_\mathrm{R}$ symmetry

- No global symmetries, all marginal deformations are exactly marginal
- Extra global symmetry $G$, exactly marginal couplings are given by a quotient

$$\mathcal{M}_\mathrm{c} = \{\lambda_i\} /\!\!/ G$$

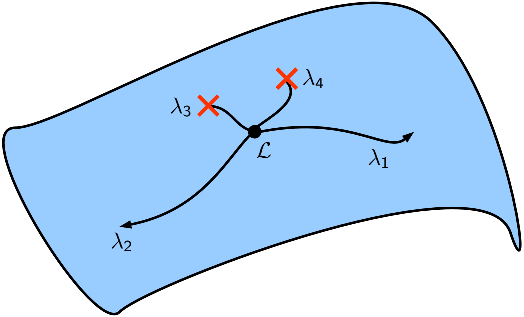
$\mathcal{N} = 4$ SYM has SU(4) R-symmetry – $\mathrm{SU}(3) \times \mathrm{U}(1)_\mathrm{R} \subset \mathrm{SU}(4)$

$$\mathcal{M}_\mathrm{c} = \{f^{ijk}\} /\!\!/ \mathrm{SU}(3) \qquad \dim \mathcal{M}_\mathrm{c} = 10 - 8 = 2$$

# General prescription

$\mathcal{N} = 1$ SCFT $\mathcal{L}$ with global symmetry $G$

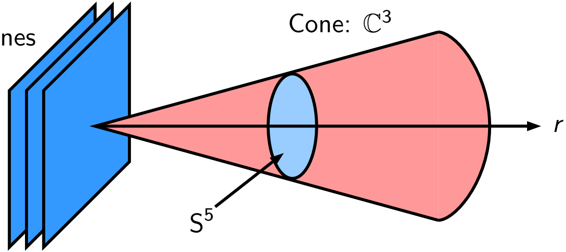
- No Kähler deformations – only superpotential
- Deform by marginal operators $\lambda_i \mathcal{O}^i$
- Some marginal directions are obstructed
- Unobstructed directions given by $\mathcal{M}_{\mathrm{c}} = \{\lambda_i\} /\!/ G$



# AdS / CFT

$\mathcal{N} = 1$ SCFT dual to type IIB supergravity on $\mathrm{AdS}_5 \times M_5$ with eight supercharges ($\mathcal{N} = 2$)

- $\mathcal{N} = 4$ SYM $\longleftrightarrow$ IIB on $\mathrm{AdS}_5 \times \mathrm{S}^5$
- Conformal manifold $\longleftrightarrow$ moduli space of vacua



## $\mathrm{AdS}_5 \times \mathrm{S}^5$ in IIB

$$\mathrm{d}s^2(\mathrm{S}^5) = \mathrm{d}s^2(\mathbb{CP}^2) + (\mathrm{d}\psi + \eta)^2$$

$$F_5 = \mathrm{d}\, C_4 = 4\,\mathrm{vol}(\mathrm{AdS}_5) + 4\,\mathrm{vol}(\mathrm{S}^5)$$

Killing vector: $\partial_\psi$ dual to $\sigma = \mathrm{d}\psi + \eta$

- $\mathrm{U}(1)_\mathrm{R}$ symmetry of field theory

$\mathrm{S}^5$ admits an SU(2) structure:

- Symplectic form: $\omega$; holomorphic two-form: $\Omega$

$$\mathrm{d}\omega = 0 \qquad \mathrm{d}\Omega = 3\mathrm{i}\,\sigma \wedge \Omega$$

# AdS / CFT

How do deformations appear in the dual geometry?

- Superpotential $\longleftrightarrow$ hypermultiplets
- Kähler $\longleftrightarrow$ vector multiplets

Need to understand supersymmetric flux backgrounds using these d.o.f.

# Supersymmetric backgrounds

## Questions

- What objects parametrise hypers and vectors?
- How do the deformations appear in supergravity?
- How are the exactly marginal deformations selected?
- Can we find the deformed geometries?

# Supersymmetric backgrounds

## Killing spinor equations

Supersymmetric background requires fermionic variations vanish

$$(\nabla_m \mp \tfrac{1}{8} H_{mnp}\gamma^{np})\epsilon^{\pm} + \tfrac{1}{16} \mathrm{e}^{\phi} \sum_i \not{F}_i \gamma_m \epsilon^{\mp} = 0$$

$$\gamma^m(\nabla_m \mp \tfrac{1}{24} H_{mnp}\gamma^{np} - \partial_m\phi)\epsilon^{\pm} = 0$$

## Any underlying geometry?

- Geometric structures?
- Deformations and moduli spaces?

# Well-known story in six dimensions

Spinors define invariant tensors $\omega_{mn} = \mathrm{i}\bar{\epsilon}^{+}\gamma_{mn}\epsilon^{+}$ and $\Omega_{mnp} = \bar{\epsilon}^{+}\gamma_{mnp}\epsilon^{-}$

$$
\begin{array}{ccc}
\mathrm{GL}(6;\mathbb{R}) & \supset & \mathrm{Sp}(6;\mathbb{R}) \text{ for } \omega \\
\cup & & \cup \\
\mathrm{SL}(3;\mathbb{C}) \text{ for } \Omega & \supset & \mathrm{SU}(3) \text{ for } \{\omega, \Omega\}
\end{array}
$$

Fluxes are obstruction to integrability

$$\mathrm{d}\Omega \sim \text{flux}, \qquad \mathrm{d}\omega \sim \text{flux}$$

Good for classification and new solutions, but

- Deformations are difficult, $\mathrm{d}\delta\Omega, \mathrm{d}\delta\omega \neq 0$.

[Gauntlett, Martelli, Waldram; Gauntlett, Pakis; Martelli, Sparks; Lüst, Tsimpis;...]

# The question

Keep all fluxes, warped ansatz

$$\mathrm{d}s^2_{10} = \mathrm{e}^{2\Delta}\mathrm{d}s^2(\mathrm{AdS}_5) + \mathrm{d}s^2(M_5)$$

What is the geometry of a generic $\mathcal{N} = 2$ flux background?

- Pair of objects that define geometric structure?
- Integrability?
- Moduli space?

[Graña, Louis, Sim, Waldram; Graña, Orsi; Graña, Triendl]

# Outline


1 Introduction

2 Generalised geometry

3 Generalised structures

4 Marginal deformations

# Generalised geometry

## Basic idea

Unifies diffeomorphism and gauge symmetries

- Generalised tangent bundle whose sections parametrise the symmetries.
- Generalised Lie derivative by which the symmetries act.

Focus on type IIB

- Fields $\{g, \phi, B, C^+, \Delta\}$ on internal $M_5$.

[Coimbra, Strickland-Constable, Waldram; Hull; Pacheco, Waldram; Berman, Perry;...]
cf. [Hitchin; Gualtieri; Baraglia; Cremmer, Julia; de Wit, Nicolai; Siegel; Hohm, Kwak, Zweibach; Jeon, Lee, Park;...]

# Generalised geometry

## Generalised tangent bundle

$$E \simeq TM \oplus T^*M \oplus \wedge^5 T^*M \oplus \wedge^- T^*M$$

$$V^M = (v^m, \lambda_m, \tilde{\lambda}_{m_1 \ldots m_5}, \lambda^-)$$

$E$ encodes diffeomorphisms and gauge transformations, e.g.

$$\delta B = \mathcal{L}_v B + \mathrm{d}\lambda, \qquad \delta C^+ = \mathcal{L}_v C^+ + \mathrm{d}\lambda^-$$

## Generalised Lie derivative

$$L_V = \text{diffeos} + \text{gauge} \qquad \text{"Leibniz algebroid"}$$

## Adjoint bundle

Tensors transform as $E_{d(d)} \times \mathbb{R}^+$ representations

$$\mathrm{ad}\,\tilde{F} \simeq \mathbb{R} \oplus (TM \otimes T^*M) \oplus \Lambda^2 TM \oplus \Lambda^2 T^*M \oplus \Lambda^6 TM \oplus \Lambda^6 T^*M \oplus \Lambda^+ TM \oplus \Lambda^+ T^*M$$

$$R^M{}_N = (\ldots, B_{mn}, \ldots, C^+)$$

Potentials give isomorphism between $E$ and $TM \oplus T^*M \oplus \ldots$

$$V = \mathrm{e}^{B+C^+}\tilde{V}$$

# "Supergravity = generalised geometry"

Neatly describes supergravity on $M_5$

- Generalised metric $G_{MN}$ equivalent to $\{g, \phi, B, C^+, \Delta\}$.

Analogue of Levi-Civita connection

- Gen. torsion-free connection $D$, compatible with gen. metric: $DG = 0$.

Gen. Ricci tensor gives bosonic action

$$S_{\mathrm{B}} = \int_{M_5} |\mathrm{vol}_G|\, R \quad \Longrightarrow \quad \text{eq. of motion} = \text{gen. Ricci flat}$$

[Coimbra, Strickland-Constable, Waldram]

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# Generalised structures

Spinors define invariant tensors

$$J_\alpha = \mathrm{e}^{B+C^\pm+\cdots}(\sigma_\alpha^{ij}\,\epsilon_i \otimes \bar{\epsilon}_j), \qquad K = \mathrm{e}^{B+C^\pm+\cdots}(\epsilon^{ij}\,\epsilon_i \otimes \epsilon_j^{\mathrm{T}}).$$

Together they define a generalised USp(6) structure

$$\begin{array}{ccc} \mathrm{E}_{6(6)} \times \mathbb{R}^+ & \supset & \mathrm{F}_{4(4)} \text{ for } K \\ \cup & & \cup \\ \mathrm{SU}^*(6) \text{ for } J_\alpha & \supset & \mathrm{USp}(6) \text{ for } \{J_\alpha, K\} \end{array}$$

# Generalised structures

## H structure

$$J_\alpha \in \Gamma(\mathrm{ad}\,\tilde{F} \otimes (\det T^*M)^{1/2})$$

Tensor in **78** of $E_{6(6)} \times \mathbb{R}^+$ giving $\mathfrak{su}_2$ algebra

$$[J_\alpha, J_\beta] = 2\kappa\epsilon_{\alpha\beta\gamma} J_\gamma, \qquad \mathrm{tr}(J_\alpha J_\beta) = -\kappa^2 \delta_{\alpha\beta} \quad \in \Gamma(\det T^*M)$$

## V structure

$$K \in \Gamma(E) \quad \text{satisfying } c(K, K, K) \neq 0$$

Vector in **27** of $E_{6(6)} \times \mathbb{R}^+$ where $c$ is the $E_{6(6)}$ cubic invariant.

# Compatibility and USp(6)

## HV structure

The structures are compatible if

$$J_\alpha \cdot K = 0, \qquad \mathrm{tr}(J_\alpha J_\beta) = -c(K,K,K)\delta_{\alpha\beta}$$

Structures intersect on $\mathrm{SU}^*(6) \cap \mathrm{F}_{4(4)} = \mathrm{USp}(6)$.

$$\text{Compatible pair } \{J_\alpha, K\} \quad \Longleftrightarrow \quad \text{USp(6) structure}$$

$$K \sim \mathrm{e}^{C_4}(\xi + \sigma \wedge \omega), \quad J_+ \sim \mathrm{e}^{C_4}(\Omega + \Omega^\sharp)$$

$$J_\alpha \cdot K = 0 \quad \longrightarrow \quad \omega \wedge \Omega = \imath_\xi \omega = \imath_\xi \Omega = 0$$

# Supersymmetry $\Longleftrightarrow$ Integrability

## Supersymmetry

Integrability for $\{J_\alpha, K\}$ is

$$\mu_\alpha(V) \coloneqq -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\int \mathrm{tr}(J_\beta L_V J_\gamma) = \lambda_\alpha \int c(K, K, V),$$

$$L_K K = 0, \qquad L_K J_\alpha = \epsilon_{\alpha\beta\gamma}\lambda_\beta J_\gamma$$

These are equivalent to solving the Killing spinor equations

> Integrable $\{J_\alpha, K\} \Longleftrightarrow \mathcal{N} = 2$ flux background

# Integrability for H structures

Consider space of H structures, coordinates $J_\alpha \in \mathcal{A}_\mathrm{H}$

- $\mathcal{A}_\mathrm{H}$ has hyper-Kähler metric, inherited fibrewise from

$$J_\alpha(x) \in \frac{\mathrm{E}_{6(6)} \times \mathbb{R}^+}{\mathrm{SU}^*(6)}$$

- Hyper-Kähler structure on $\mathcal{A}_\mathrm{H}$ preserved by diffeos and gauge transformations, parametrised by $V \in \Gamma(E) \simeq \mathfrak{gdiff}$

$$\delta J_\alpha = L_V J_\alpha \quad \in T\mathcal{A}_\mathrm{H}$$

Moment maps for action of GDiff

$$\mu_\alpha(V) = -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma} \int \mathrm{tr}(J_\beta L_V J_\gamma)$$

# Integrability for H structures

## Moduli space

Structures related by GDiff are equivalent, moduli space is a hyper-Kähler quotient

$$\mathcal{M}_{\mathrm{H}} = \mathcal{A}_{\mathrm{H}} /\!/\!/ \mathrm{GDiff}$$

$L_K J_\alpha = \epsilon_{\alpha\beta\gamma}\lambda_\beta J_\gamma$ takes a Kähler slice – Kähler quotient

- Not surprising c.f. gauged supergravity
- Quotient suggests same structure as dual field theory
- How do global symmetries appear?
- What has all this formalism bought for you?

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# Kähler deformations

### $\delta J_\alpha = 0$ and $\delta K \neq 0$ – Kähler deformations

Dual to Kähler deformations

$$L_{\delta K} J_\alpha = 0 \quad \Longrightarrow \quad \delta\mu_\alpha(V) = \lambda_\alpha \int c(\delta K, K, V) = 0$$

but $\delta K = 0$ is only solution

No Kähler deformations

# Superpotential deformations

### $\delta J_\alpha \neq 0$ and $\delta K = 0$ – superpotential deformations

Solutions to $\delta\mu_\alpha = 0$ and $L_K \delta J_\alpha = 0$ are marginal deformations

- Gives infinitesimal solution – equivalent to turning on three-form fluxes

Infinitesimal solution can be extended unless there are obstructions

- Which deformations extend to all orders?
- Where are the global symmetries?

# Higher-order deformations

## Shortcut

Obstructions are conditions missed by the moment maps

- Any $V$ such that $L_V J_\alpha = 0$ satisfies the moment maps trivially
- $L_V J_\alpha = L_V K = 0$ imply $V$ corresponds to a Killing vector that commutes with $\xi$
- $\{V\}$ define $G \times \mathrm{U}(1)_\mathrm{R} \subset \mathrm{Iso}(M_5)$ – global symmetry!

Impose missing conditions using moment map for $G$

- Missing equations given by quotient by $G$ on space of linearised deformations

# Example: $\mathrm{AdS}_5 \times \mathrm{S}^5$

## Marginal deformations

$\delta J_\alpha$ generates flux, dual to marginal deformations $\lambda_i \mathcal{O}^i$

- $\delta\mu_\alpha(V) = 0$ fixes flux in terms of function $f$ which is holomorphic on cone

$$F_3 + \mathrm{i}\,H \propto f\sigma \wedge \bar{\Omega} + \dots \qquad \bar{\partial} f = 0$$

- $L_K J_\alpha$ fixes charge of $f$

$$\mathcal{L}_\xi f = 3\mathrm{i}\, f$$

$\mathbb{C}^3$ with coordinates $z_i$: $\mathcal{L}_\xi z_i = \mathrm{i} z_i$

$$f = f^{ijk} z_i z_j z_k$$

# Example: $\mathrm{AdS}_5 \times S^5$

## Obstruction from global symmetry

Higher-order calculations constrain $f$ – long and difficult! [Aharony, Kol, Yankielowicz]

$S^5$ has $SO(6) \cong SU(4)$ isometry with $SU(3) \times U(1)_R$ subgroup

- Obstruction due to SU(3) that preserves $\{J_\alpha, K\}$
- Missing conditions are moment map for SU(3)

$$\gamma^i{}_j = f^{ikl}\bar{f}_{jkl} - \tfrac{1}{3}\delta^i{}_j f^{klm}\bar{f}_{klm} = 0$$

- Reproduces beta function from field theory. Only $10 - 8 = 2$ complex degrees of freedom in $f$.

# Summary

Generalised geometry is a natural language for supersymmetric backgrounds

- Flux backgrounds characterised by generalised structures
- Generalised structures package hyper and vector d.o.f. – same as dual field theory

Supergravity realisation of classic field theory result

- Can find infinitesimal deformations and exactly marginal deformations from obstructions
- Works for $AdS_5$ and $AdS_4$ backgrounds

# Future work

## Finite deformations

- Known for Lunin–Maldacena deformations: $J_\alpha \to e^\beta J_\alpha$
- Metric on conformal manifold?

## Dual quantities of field theory

- Central charge: $a^{-1} \sim \int c(K, K, K)$
- $a$-maximisation as variational problem?
- Dimension of operators from wrapped branes

## Topological theories