Calabi-Yau metrics, machine learning and the spectrum of the Laplace operator

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Motivation

Does string theory describe our universe?

- Heterotic string on Calabi–Yau comes closest to realistic MSSM models

Usually focus on getting correct gauge group, matter spectrum, superpotential, etc.

- Do not need details of metric for these

How many of these string vacua are physically reasonable?

String theory

Problem: a ‘theory of everything’ should give particle masses, couplings, supersymmetry breaking patterns

- Need Kähler potential of theory and zero modes – depend on the metric on the internal space
- No explicitly known (compact) Calabi–Yau metrics!

This talk: computing these metrics numerically and progress on zero modes

Outline

Review of $\mathcal{N} = 1$ compactifications

Numerical Calabi–Yau metrics

Application: the spectrum of the Laplace operator

Review of $\mathcal{N} = 1$ compactifications

Heterotic string theory

Low-energy limit is 10d supergravity coupled to Yang–Mills

Want Minkowski compactifications that preserve some supersymmetry

$$M_{10} = \mathbb{R}^{1,3} \times X$$

X is 6d and compact with vector bundle V

- Metric g
- Dilaton ϕ
- Gauge fields A with $G \subseteq \mathrm{E}_8 \times \mathrm{E}_8$
- 3-form flux H

Compactification

Minimal ($N = 1$) SUSY in 4d requires SU(3) holonomy [Candelas et al. '85]

- No H flux
- X is Calabi–Yau

Physics in 4d determined by geometry of X – Kaluza–Klein reduction fixes 4d modes

- e.g. for scalars, masses in 4d c.f. eigenvalues of Laplacian in 6d

$$\Box_{10}(\zeta_4 \otimes \phi_6) = 0 \equiv \Box_4 \zeta_4 \otimes \phi_6 - \zeta_4 \otimes \Delta_6 \phi_6$$

$$\Delta_6 \phi_6 = \lambda \phi_6 \quad \Rightarrow \quad \Box_4 \zeta_4 = \lambda \zeta_4 \equiv m^2 \zeta_4$$

Zero modes $(\lambda = 0)$ determine low-energy physics

Physics from Calabi–Yaus

Particle content comes from choice of X and V, e.g.

- $\mathrm{SU}(3)$ bundle gives E_6 gauge group in 4d, further broken to SM group by flat bundle
- $\frac{1}{2}\chi(X) = 3$ gives correct number of particle generations
- Many top-down models with promising MSSM-like spectrums

Masses of quarks and leptons from cubic Yukawa couplings

- Come from triple overlap of zero modes on X coupled to gauge field

$$C_{abc} = \int_X \psi_a \cdot \psi_b \cdot \psi_c \sqrt{g}\mathrm{d}^6x$$

- Often cohomological calculations but need ψ_a to be normalised zero modes

A wish list

Zero modes $\lambda = 0$ give light particles in 4d ✓

- Zero modes reduce to cohomology

Yukawa couplings ✗

- Cohomology calculation but missing normalisation $\int_X |\psi_a|^2 \sqrt{g} \mathrm{d}^6 x = 1$

Supersymmetry breaking ✗

- Soft masses and couplings c.f. $N = 1$ Kähler potential and normalised zero modes [Kaplunovsky, Louis '93; Blumenhagen et al. '09; ...]

Massive modes $\lambda > 0$ ✗

- Extra low-lying modes? At what scale? Swampland distance conjecture

Numerical Calabi–Yau metrics

What is a Calabi–Yau?

Calabi–Yau manifolds are (complex) Kähler manifolds with a Ricci-flat metric

- Kähler with $c_1(X) = 0 \Rightarrow$ there exists a Ricci-flat metric [Yau '77]
- Existence but no explicit constructions

Kähler $\Rightarrow$ Kähler potential K gives (real) two-form $J = \partial\bar{\partial}K$ s.t.

$$J^3 = \mathrm{vol}_J \quad \text{and} \quad \mathrm{d}J = 0$$

$c_1(X) = 0 \Rightarrow$ (complex) nowhere-vanishing $(3,0)$-form Ω s.t.

$$|\Omega|^2 = \mathrm{vol}_\Omega \quad \text{and} \quad \mathrm{d}\Omega = 0$$

Example: Fermat quintics

Quintic hypersurface X in $\mathbb{P}^4$ with $(z_0, \ldots, z_4) \sim t(z_0, \ldots, z_4)$

$$Q(z) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi\, z_0 z_1 z_2 z_3 z_4 = 0$$

$c_1(X) = 0 \Rightarrow$ three-form Ω fixed by $Q(z)$, e.g. in $z_0 = 1$ patch

$$\Omega = \frac{\mathrm{d}z_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}z_4}{\partial Q / \partial z_1}$$

Metric (and J) completely determined by Kähler potential

$$g_{i\bar{j}}(z, \bar{z}) = \partial_i \bar{\partial}_{\bar{j}} K(z, \bar{z}), \qquad \mathrm{vol}_J \sim \det g_{i\bar{j}} \mathrm{d}^6 z$$

How do we measure accuracy?

How do we know whether some $g_{i\bar{j}}$ is the Calabi–Yau metric?

The Ricci-flat metric is given by a K that satisfies (c.f. Monge–Ampère)

$$\left.\frac{\mathrm{vol}_J}{\mathrm{vol}_\Omega}\right|_p = \text{constant} \quad \Rightarrow \quad R_{i\bar{j}} = 0$$

The volumes are easier to calculate than the Ricci tensor. Normalising the volumes we define a functional of K

$$\sigma(K) = \int_X \left| 1 - \frac{\mathrm{vol}_\Omega}{\mathrm{vol}_J} \right| \mathrm{vol}_\Omega$$

The exact CY metric has $\sigma = 0$

Numerical metrics

The problem of finding the Ricci-flat metric on a Calabi–Yau then reduces to finding a single function $K(z, \bar{z})$ that minimises σ

There are many approaches to this problem:

- Position space methods [Headrick, Wiseman '05]
- Spectral methods [Donaldson '05; Douglas et al. '06; Braunet al. '07; Headrick, Nassar '09]
- Regression methods [AA et al. '19]
- Neural networks [Douglas et al. 20; Anderson et al. '20]

One can also try to find $g_{i\bar{j}}(z, \bar{z})$ directly (but need to impose Kähler, overlap conditions, etc.) [Anderson et al. '20; Jejjala '20]

Donaldson's ansatz

Natural Kähler metric on $\mathbb{P}^4$ given by

$$K_{\mathrm{FS}} = \log \sum_{i=0}^{4} z_i \bar{z}_{\bar{i}}$$

Can generalise this with a hermitian matrix $h^{i\bar{j}}$

$$K(h) = \log \sum_{i,\bar{j}=0}^{4} z_i h^{i\bar{j}} \bar{z}_{\bar{j}}$$

Restricting to $X \subset \mathbb{P}^4$ (defined by $Q(z) = 0$) gives a Kähler metric but not Ricci-flat

- 25 real parameters in $h^{i\bar{j}}$ that we can vary
- Need more parameters to better approximate the Ricci-flat metric

Donaldson's ansatz

Generalise by replacing coordinates z_i with homogeneous polynomials s_α of degree k

$$\text{e.g.} \quad k = 2: \quad s_\alpha = (z_0^2, z_0 z_1, z_0 z_2, \ldots)$$

Kähler potential is then

$$K(h) = \log \sum_{\alpha,\bar\beta=0}^{14} s_\alpha h^{\alpha\bar\beta} \bar{s}_{\bar\beta}, \qquad h^{\alpha\bar\beta} \sim 225 \text{ parameters}$$

At degree k have $N_k \sim \mathcal{O}(k^3)$ parameters, so can approximate the Ricci-flat metric to arbitrary precision!

- 'Algebraic metrics' – higher k allows better precision (smaller σ)
- Spectral method as $s_\alpha \bar{s}_{\bar\beta}$ give a basis for eigenspaces of Laplacian on $\mathbb{P}^4$ ($\mathrm{U}(1)$-invariant spherical harmonics on S^9)

How to fix $h^{\alpha\bar{\beta}}$?

Finding the 'best' approximation to the Ricci-flat metric on X amounts to finding $h^{\alpha\bar{\beta}}$ so that σ is minimised

Three approaches:

- Iterative procedure [Donaldson '05; Douglas '06; Braun '07]
- Minimise σ directly (Mathematica) [Headrick, Nassar '09]
- Treat σ as a loss function for a neural network [Douglas et al. 20; Anderson et al. '20]

In all cases, numerical integrals carried out by Monte Carlo

Donaldson's algorithm [Donaldson '05]

Approximate the Ricci-flat metric by the 'balanced' metric on X

- Define the T operator as

$$T(h)_{\alpha\bar{\beta}} = \int \mathrm{vol}_\Omega \frac{s_\alpha \bar{s}_{\bar{\beta}}}{\sum_{\gamma,\bar{\delta}} s_\gamma h^{\gamma\bar{\delta}} \bar{s}_{\bar{\delta}}},$$

- $h^{\alpha\bar{\beta}}$ is 'balanced' when $h^{\alpha\bar{\beta}} = \left[T(h)_{\alpha\bar{\beta}}\right]^{-1}$

As $k \to \infty$, the balanced metrics $K(k)$ converge to the exact Calabi–Yau metric (also works for Einstein metrics $R_{i\bar{j}} \propto g_{i\bar{j}}$).

- How do you solve for the balanced $h^{\alpha\bar{\beta}}$? Iterate using

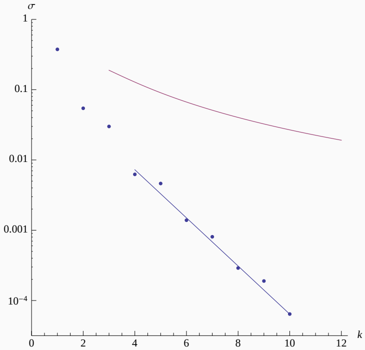
$$h^{\alpha\bar{\beta}}_{(n+1)} = \left[T(h_{(n)})_{\alpha\bar{\beta}}\right]^{-1}$$

The fixed point is the balanced metric (guaranteed convergence!)

Optimal metrics [Headrick, Nassar '09]

The '**optimal metrics**' are the most precise. For Fermat quintic

$$\sigma_{\text{Donaldson}} \sim k^{-2}, \quad \sigma_{\text{optimal}} \sim 2.2^{k}$$



(c.f. [Headrick, Nassar '09])

Problem: number of parameters grows as k^6 – '**curse of dimensionality**' – and exponentially slower

- The optimal metrics work well only for small k or where there is a **large symmetry** that reduces the number of independent parameters

Solution: machine learning is well suited to these kinds of problems

Initial question: is the data of a Calabi–Yau metric 'learnable'? [Ashmore et al. '19]

Consider trying to learn $\det g_{i\bar{j}}$ via supervised learning

- Compute $\det g_{i\bar{j}}^{(k)}$ for 10,000 points on X at degrees $k = (4, 5, 6, 7)$
- Extrapolate to $\det g_{i\bar{j}}^{(k)}$ for larger k, e.g. $k = 12$

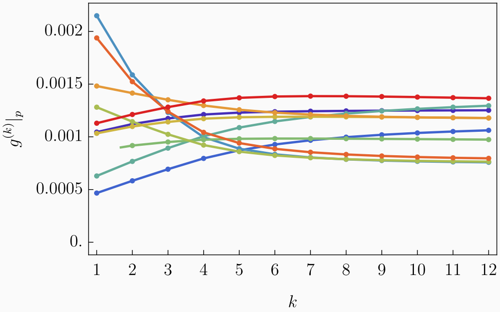
We then have labelled data $\{\text{Inputs} \to \text{Outputs}\}$ of the form

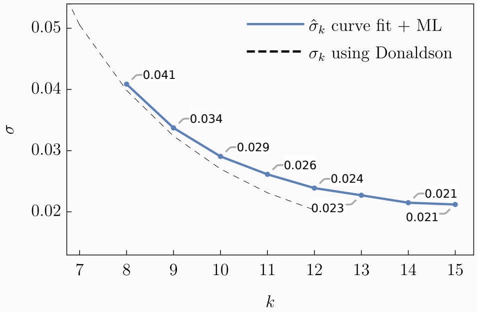
$$\text{Inputs} = \left\{ p = (z_0, \ldots, z_4), \det g_{i\bar{j}}^{(4)}\Big|_p, \ldots, \det g_{i\bar{j}}^{(7)}\Big|_p \right\}, \qquad \text{Outputs} = \left\{ \det g_{i\bar{j}}^{(k)}\Big|_p \right\}$$

Machine learning

Feed this into a 'gradient-boosted decision tree' – result is a function that takes inputs from an unseen point on X and gives the predicted value of $\det \hat{g}^{(k)}_{i\bar{j}}\Big|_p$

- We can then compare the σ accuracy to test how well we are doing





Similar σ values – but approx. two orders of magnitude quicker than direct $k = 12$ calculation!

State of the art [Anderson et al. '20]

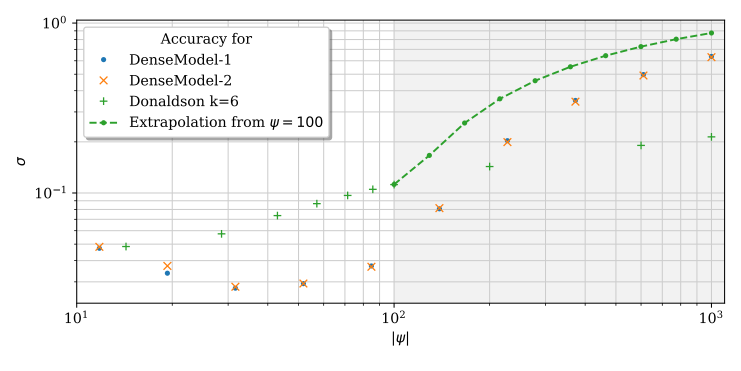
Learning $h^{\alpha\bar{\beta}}$ including dependence on complex structure ψ using neural network

- For phenomenology, want to be able to scan over values of moduli / understand how couplings and masses depend on moduli
- Loss function is $\sigma(K)$ – minimise by stochastic gradient descent
- Uses ψ as input feature
- Single dense hidden layer of width $\dim\{s_\alpha\}$ or $(\dim\{s_\alpha\})^2$ (computing K at degree k)

$$\begin{array}{ccccccccc} \psi & \to & \text{Model} & \to & h^{\alpha\bar{\beta}} & \to & K(z,\bar{z}) & \to & g_{i\bar{j}} \\ & & & & & & \uparrow & & \\ & & & & & & z_i & & \end{array}$$

State of the art [Anderson et al. '20]

Results when trained on $\psi \in (0, 100)$ at $k = 6$ (c.f. [Anderson et al. '20])



Gives precision comparable to Donaldson at $k = 12$

- Donaldson at $k = 12$ takes days to run over the range $\psi = 0, \ldots, 100$ – neural network trained in minutes for same range
- Gives full dependence on complex structure moduli!

State of the art [Douglas et al. 20]

Recall that the CY metric is determined by K via

$$K = \log(s_\alpha h^{\alpha\bar{\beta}} \bar{s}_{\bar{\beta}}),$$

where s_α are homogeneous, degree-k polynomials of the z_i

Replace this with a feed-forward network $F(z, \bar{z})$ such that $K = \log F$

$$\begin{aligned}(\mathrm{re}\, z_i \bar{z}_j, \mathrm{im}\, z_i \bar{z}_j) &\mapsto F(z, \bar{z}) \\ F &= \theta_d \circ W^{(d)} \ldots \circ \theta_1 \circ W^{(1)}\end{aligned}$$

where $W^{(n)}$ are linear maps (weights) and $\theta_n(x) = x^2$ are non-linear homogeneous activation functions

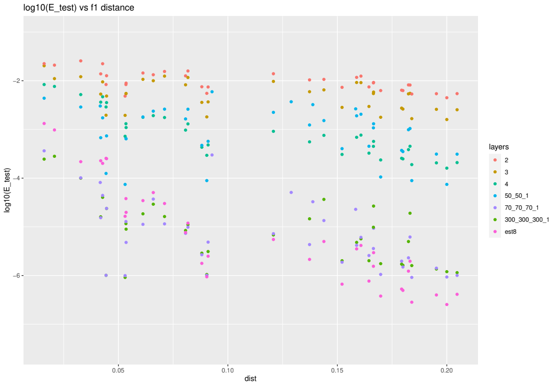
- Number of parameters $= 25 \times \text{width}_1 + \text{width}_1 \times \text{width}_2 + \ldots + \text{width}_d \times 1$

State of the art [Douglas et al. 20]

Has the advantage that you do not fill out the space of metrics at degree k

- Can resolve features at scales k^{-1} with degree k polynomial basis
- Use a network with smaller width but more layers (depth) $\Rightarrow$ higher effective k – 11,620 parameters for $(70, 70, 70)$ network vs 245,025 for $k = 8$

σ values for $k = 2, 3, 4$ optimal and various networks (c.f. [Douglas et al. 20])



Application: the spectrum of the Laplace operator

The Laplace operator [AA '20]

Important phenomenological details of models determined by harmonic modes on CY

Ignoring gauge sector, harmonic modes are (p, q)-eigenforms of the Laplace operator

$$\Delta = \mathrm{d}\delta + \delta\mathrm{d}, \qquad \Delta|\phi_n\rangle = \lambda_n|\phi_n\rangle$$

where λ_n are real and non-negative and appear with multiplicity μ_n (c.f. continuous or finite symmetries)

- Need some way of finding both the spectrum and the harmonic modes themselves
- Scalar case done [Braun et al. '08]

The Laplace operator

For each (p,q), given a (non-orthonormal) basis of (p,q)-forms $\{\alpha_A\}$ we can expand the eigenmodes as

$$|\phi\rangle = \sum_A \langle \alpha_A | \tilde{\phi} \rangle \, |\alpha_A\rangle$$

so that

$$\begin{aligned} \Delta|\phi\rangle &= \lambda|\phi\rangle \\ \sum_B \langle \alpha_A|\Delta|\alpha_B\rangle\langle\alpha_B|\tilde{\phi}\rangle &= \sum_B \lambda\langle\alpha_A|\alpha_B\rangle\langle\alpha_B|\tilde{\phi}\rangle \\ \Rightarrow \quad \Delta_{AB}\tilde{\phi}_B &= \lambda O_{AB}\tilde{\phi}_B \end{aligned}$$

Generalised eigenvalue problem for λ and $\tilde{\phi}_A$

The Laplace operator

Basis $\{\alpha_A\}$ is infinite dimensional – truncate to a finite approximate basis at degree k_ϕ in z_i

$$\frac{(\text{degree } k_\phi \ (p,0)\text{-form})(\text{degree } k_\phi \ (0,q)\text{-form})}{(|z_0|^2 + \ldots |z_4|^2)^{k_\phi}}$$

where we have (c.f. harmonic forms on $\mathbb{P}^4$)

$$\{\text{degree } k_\phi \ (0,0)\text{-form}\} = \text{degree } k_\phi \text{ polynomials}$$

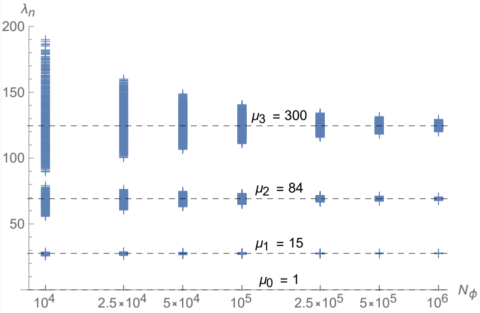
$$\{\text{degree } 2 \ (1,0)\text{-form}\} = \{z_0 dz_1 - z_1 dz_0, z_0 dz_2 - z_2 dz_0\}$$

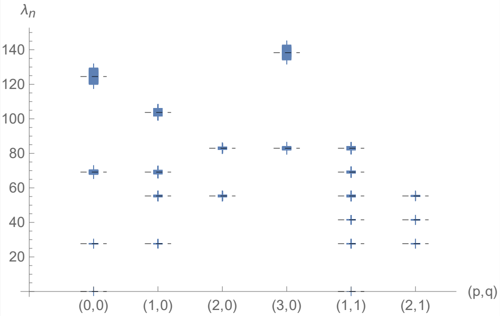
$$\vdots = \vdots$$

1. Hodge star and complex conjugation $\Rightarrow$ $\lambda_n^{(p,q)} = \lambda_n^{(q,p)} = \lambda_n^{(3-p,3-q)} = \lambda_n^{(3-q,3-p)}$
2. Compute matrices Δ_{AB} and O_{AB} numerically for independent choices of (p,q)
3. Find eigenvalues and eigenvectors

Example: $\mathbb{P}^3$ [Ikeda, Taniguchi '78]

Numeric results: spectrum on $\mathbb{P}^3$ at $k_\phi = 3$ with 10^6 points for numerical integrals, use exact metric on $\mathbb{P}^3$ – $(0,0)$ spectrum for varying number of points and (p,q) spectrum





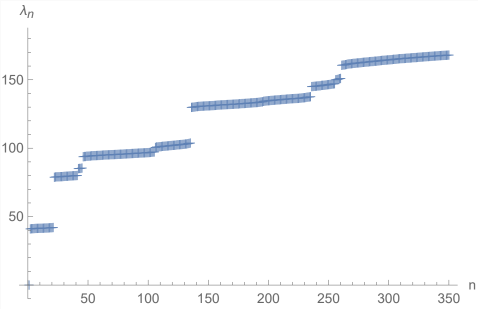
- Eigenvalues and multiplicities determined by SU(4) representations (set Vol $= 1$)
- Degeneracy of eigenvalues recovered as number of integration points $\to \infty$ (restores SU(4) symmetry)

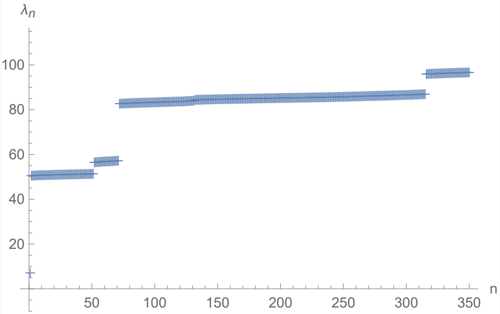
Fermat quintic

On CY we don't have the exact metric:

1. Specify the CY (e.g. Fermat quintic) and compute metric numerically
2. Pick a finite basis for (p, q)-forms at some degree
3. Solve numerically for eigenvalues and eigenvectors of Laplace operator (for each choice of (p, q))

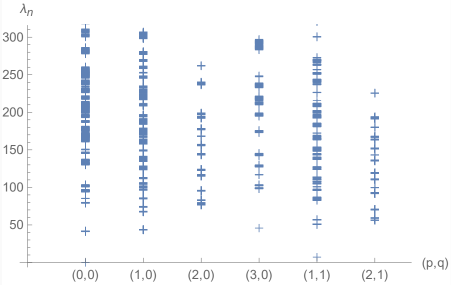
Spectrum for $(0, 0)$- and $(1, 1)$-forms





Fermat quintic

(p,q)	$(0,0)$		$(1,0)$		$(2,0)$	
$\dim\{\alpha_A\}$	1225		1400		350	
n	λ_n	μ_n	λ_n	μ_n	λ_n	μ_n
0	0.00	1	43.7 ± 0.2	20	77.0 ± 0.3	30
1	41.5 ± 0.3	20	67.7 ± 0.3	30	79.0 ± 0.4	30
2	79.4 ± 0.4	20	74.1 ± 0.3	30	83.1 ± 0.3	20
3	85.3 ± 0.2	4	85.4 ± 0.4	34*	95.5 ± 0.3	20
4	95.5 ± 0.9	60	96.8 ± 0.4	20	116 ± 1	40
5	102 ± 1	30	101 ± 1	60	123 ± 1	30
(p,q)	$(3,0)$		$(1,1)$		$(2,1)$	
$\dim\{\alpha_A\}$	350		1600		400	
n	λ_n	μ_n	λ_n	μ_n	λ_n	μ_n
0	45.8	1	7.13	1	56.4 ± 0.1	20
1	98.9 ± 0.4	20	50.8 ± 0.2	30	59.2 ± 0.2	20
2	103 ± 0.4	20	51.2 ± 0.1	20	70.5 ± 0.2	30
3	117 ± 0.2	4	56.9 ± 0.2	20	92.3 ± 0.4	60
4	128 ± 1	30	83.3 ± 0.4	60	99.7 ± 0.2	4
5	144 ± 1	30	85.5 ± 0.7	184*	111 ± 0.4	40



Ω and J should give $\lambda = 0$ eigenmodes for $(3,0)$ and $(1,1)$ – improves as size of approximate basis is increased

Multiplicities = dimension of irreps of $(S_5 \times \mathbb{Z}_2) \ltimes (\mathbb{Z}_5)^4$

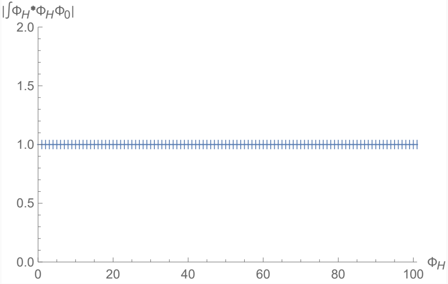
Yukawa couplings

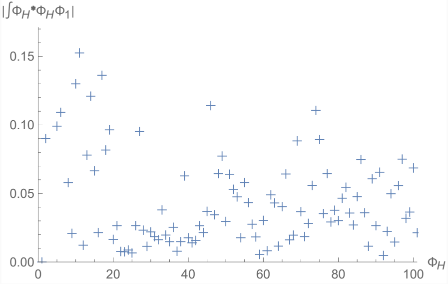
Honest Yukawa couplings computed by $(0, q)$-forms valued in gauge bundle V

$$W = \ldots + \lambda_u QHu + \ldots, \qquad \lambda_u = \int_X \Omega \wedge \mathrm{tr}(\Psi_Q \wedge \Psi_H \wedge \Psi_u)$$

Toy example: triple overlap of a light scalar mode with two heavy scalar modes

$$\left| \int_X \bar{\Phi}_\mathrm{H} \Phi_\mathrm{H} \Phi_\mathrm{L} \right| = |Y_{\bar{\mathrm{H}}\mathrm{HL}}| \quad \text{where } \Phi_m = \frac{\phi_m}{\sqrt{\langle \phi_m | \phi_m \rangle}}$$





Summary

- Calabi–Yau metrics are important for getting real predictions from string theory
- Analytic metrics not known (and may never be) so must rely on numerical results
- Many methods to compute metrics – ML looks extremely promising for this!
- With the 'data' of the metric, one can compute eigenmodes of the Laplace operator

Outlook

- Include vector bundles (gauge fields)
    - Donaldson works for bundles too [Douglas et al. '06] – can diagnose stability
    - ML for gauge connections? Harmonic modes?
- CY threefolds appear as target spaces for $N = (2,2)$ SCFTs with $c = 9$
    - Spectrum of CY $\subset$ spectrum of CFT operators
    - Overlap integrals $\equiv$ OPEs in CFT
- SYZ conjecture? F-theory?

Thank you!