Generalising Calabi–Yau for flux backgrounds

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The question

What is the geometry of a generic $\mathcal{N} = 2$ flux background?

- No fluxes $\rightarrow$ complex/symplectic geometry.
- NS-NS fluxes $\rightarrow$ generalised complex geometry.
- All fluxes $\rightarrow$ exceptional generalised geometry.

Outline

Supersymmetric backgrounds

Backgrounds with no fluxes

$$\nabla\epsilon = 0 \quad \Longrightarrow \quad \text{special holonomy}$$

e.g. type II on Calabi–Yau

$$\omega_{mn} \sim \epsilon^{\dagger}\gamma_{mn}\epsilon, \qquad \Omega_{mnp} \sim \epsilon^{T}\gamma_{mnp}\epsilon$$

Integrable SU(3) structure if

$$\mathrm{d}\Omega = \mathrm{d}\omega = 0 \quad \Leftrightarrow \quad \nabla\epsilon = 0$$

Calabi–Yau $\Leftrightarrow$ integrable SU(3) structure $\Leftrightarrow$ $\mathcal{N} = 2$ in 4d

Supersymmetric backgrounds

Backgrounds with fluxes

With fluxes, Levi-Civita $\to$ supergravity connection, e.g. type II

$$(\nabla_m \mp \tfrac{1}{8} H_{mnp}\gamma^{np})\epsilon^{\pm} + \tfrac{1}{16} e^{\phi} \sum_i \not{F}_i \gamma_m \epsilon^{\mp} = 0$$

$$\gamma^m(\nabla_m \mp \tfrac{1}{24} H_{mnp}\gamma^{np} - \partial_m \phi)\epsilon^{\pm} = 0$$

Any underlying geometry?

- Special holonomy?
- Analogues of ω and Ω? Integrability?
- Deformations and moduli spaces?

Supersymmetric backgrounds

One approach: G-structures

Killing spinors stabilised by

$$G \subset \mathrm{SO}(6) \subset \mathrm{GL}(6)$$

and define a G-structure, but not integrable

$$\mathrm{d}\Omega \sim \text{flux}, \qquad \mathrm{d}\omega \sim \text{flux}$$

Good for classification and new solutions, but

- Global issues: "type changing".
- Moduli are difficult, $\mathrm{d}\delta\Omega, \mathrm{d}\delta\omega \neq 0$.

[Gauntlett, Martelli, Waldram; Gauntlett, Pakis; Martelli, Sparks; Lüst, Tsimpis;...]

Supersymmetric backgrounds

Generalised Calabi–Yau

Killing spinors stabilised by

$$\mathrm{SU}(3) \times \mathrm{SU}(3) \subset \mathrm{SO}(6) \times \mathrm{SO}(6) \subset \mathrm{O}(6,6) \times \mathbb{R}^+$$

and define a G-structure, integrable for NS-NS backgrounds

$$\mathrm{d}\Phi^+ = 0, \qquad \mathrm{d}\Phi^- = 0$$

Each $\Phi^\pm \in \Gamma(\wedge^\pm T^* M)$ defines $\mathrm{SU}(3,3)$ structure.

$$\Phi^+ = \mathrm{e}^{-\phi} \mathrm{e}^{-B - \mathrm{i}\omega}, \qquad \Phi^- = \mathrm{e}^{-\phi} \mathrm{e}^{-B} (\Omega_1 + \Omega_3 + \Omega_5)$$

[Hitchin; Gualtieri; Graña, Minasian, Petrini, Tomasiello]

Supersymmetric backgrounds

Generic $\mathcal{N} = 2$ backgrounds

Keep all fluxes, warped compactification

$$\mathrm{d}s^2 = \mathrm{e}^{2\Delta}\mathrm{d}s^2(\mathbb{R}^{3,1}) + \mathrm{d}s^2(M)$$

where M is 6d for type II and 7d for M-theory.

In sugra, spinors transform as **8** of SU(8): pair of Killing spinors stabilised by

$$\mathrm{SU}(6) \subset \mathrm{SU}(8) \subset \mathrm{E}_{7(7)} \times \mathbb{R}^+$$

and define a generalised SU(6) structure.

The question

What is the geometry of a generic $\mathcal{N} = 2$ flux background?

- Generalisations of ω and Ω? Define generalised SU(6) structure.
- Integrability?
- Moduli space?

Structures in generalised geometry

[Graña, Louis, Sim, Waldram; Graña, Orsi; Graña, Triendl]

Outline

Generalised geometry

$E_{d(d)} \times \mathbb{R}^+$ generalised geometry

Unifies all symmetries, restricted to M_{d-1} in type II or M_d in M-theory

What do we need?

- Generalised tangent bundle whose sections parametrise the symmetries.
- Generalised Lie derivative by which the symmetries act.

Focus on type II

- Fields $\{g, \phi, B, \tilde{B}, C^{\pm}, \Delta\}$ on M_{d-1}.

[Coimbra, Strickland-Constable, Waldram; Hull; Pacheco, Waldram; Berman, Perry;...]
cf. [Hitchin; Gualtieri; Baraglia; Cremmer, Julia; de Wit, Nicolai; Siegel; Hohm, Kwak, Zweibach; Jeon, Lee, Park;...]

Generalised geometry

Generalised tangent bundle

$$E \simeq TM \oplus T^*M \oplus \wedge^5 T^*M \oplus \wedge^{\pm} T^*M \oplus (T^*M \otimes \wedge^6 T^*M)$$

$$V^M = (v^m, \lambda_m, \tilde{\lambda}_{m_1\dots m_5}, \lambda^{\pm}, \tau_{m,n_1\dots n_7})$$

E encodes diffeomorphisms and gauge transformations, e.g.

$$\delta B = \mathcal{L}_v B + \mathrm{d}\lambda, \qquad \delta C^{\pm} = \mathcal{L}_v C^{\pm} + \mathrm{d}\lambda^{\pm}$$

Generalised Lie derivative

$$L_V = \text{diffeos} + \text{gauge} \qquad \text{"Leibniz algebroid"}$$

[Hull; Pacheco, Waldram; Coimbra, Strickland-Constable, Waldram]

Adjoint bundle

Tensors transform as $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ representations

$$\mathrm{ad}\,\tilde{F} \simeq \mathbb{R} \oplus (TM \otimes T^*M) \oplus \wedge^2 TM \oplus \wedge^2 T^*M \oplus \wedge^6 TM \oplus \wedge^6 T^*M$$
$$\oplus \wedge^{\pm} TM \oplus \wedge^{\pm} T^*M$$

$$R^M{}_N = (\ldots, B_{mn}, \ldots, C^{\pm})$$

Potentials give isomorphism between E and $TM \oplus T^*M \oplus \ldots$

$$V = \mathrm{e}^{B + C^{\pm}} \tilde{V}$$

Generalised geometry

"Supergravity = generalised geometry"

Neatly describes supergravity on M_{d-1}

- Generalised metric G_{MN} equivalent to $\{g, \phi, B, \tilde{B}, C^{\pm}, \Delta\}$.

Analogue of Levi-Civita connection

- Gen. torsion-free connection D, compatible with gen. metric: $DG = 0$.

Gen. Ricci tensor gives bosonic action

$$S_{\mathrm{B}} = \int_M |\mathrm{vol}_G|\, R \quad \Longrightarrow \quad \text{eq. of motion} = \text{gen. Ricci flat}$$

[Coimbra, Strickland-Constable, Waldram]

Outline

Conventional G-structures

G-structure defined by invariant tensors.

$$
\begin{aligned}
\text{symplectic} \quad \omega &: \quad G = \mathrm{Sp}(3;\mathbb{R}) \\
\text{complex} \quad \Omega &: \quad G = \mathrm{SL}(3;\mathbb{C}) \\
\{\omega, \Omega\} &: \quad G = \mathrm{SU}(3)
\end{aligned}
$$

Analogues with flux?

Generalised structures

Generalised G-structures, defined by invariant generalised tensors.

$$
\begin{aligned}
\text{H structure} \quad & J_\alpha: \quad & G = \mathrm{SO}^*(12) \\
\text{V structure} \quad & K: \quad & G = \mathrm{E}_{6(2)} \\
\text{HV structure} \quad & \{J_\alpha, K\}: \quad & G = \mathrm{SU}(6)
\end{aligned}
$$

- J_α and K associated with hyper- and vector-multiplets.
- Interpolate between symplectic, complex, product and hyper-Kähler.

[Graña, Louis, Sim, Waldram]

Hypermultiplet structures

H structure: $SO^*(12)$

Weighted tensor in $\mathbf{133_1}$ of $E_{7(7)} \times \mathbb{R}^+$

$$J_\alpha \in \Gamma(\mathrm{ad}\,\tilde{F} \otimes (\det T^*M)^{1/2})$$

giving highest weight $\mathfrak{su}_2$ algebra

$$[J_\alpha, J_\beta] = 2\kappa\epsilon_{\alpha\beta\gamma}J_\gamma, \qquad \mathrm{tr}(J_\alpha J_\beta) = -\kappa^2\delta_{\alpha\beta} \quad \in \Gamma(\det T^*M)$$

Constructed from spinor bilinears: $J_\alpha = e^{B+C^\pm+\cdots}(\sigma_\alpha^{ij}\,\epsilon_i \otimes \bar{\epsilon}_j)$

Generalises Ω in IIA and ω in IIB

Vector-multiplet structures

V structure: $E_{6(2)}$

Generalised vector in $\mathbf{56_1}$ of $E_{7(7)} \times \mathbb{R}^+$

$$K \in \Gamma(E) \quad \text{satisfying } q(K) > 0$$

where q is the $E_{7(7)}$ quartic invariant

- $q(K)$ as "Hitchin functional" defines second vector $\hat{K}$.

Constructed from spinor bilinears: $K = e^{B+C^{\pm}+\cdots}(\epsilon^{ij}\,\epsilon_i \otimes \epsilon_j^{T})$

Generalises ω in IIA and Ω in IIB

Compatibility and SU(6)

HV structure

The structures are compatible if

$$J_\alpha \cdot K = 0, \qquad \mathrm{tr}(J_\alpha J_\beta) = -2\sqrt{q(K)}\delta_{\alpha\beta}$$

analogues of $\omega \wedge \Omega = 0$ and $\frac{1}{6}\omega^3 = \frac{\mathrm{i}}{8}\Omega \wedge \bar{\Omega}$.

Structures intersect on $\mathrm{SO}^*(12) \cap \mathrm{E}_{6(2)} = \mathrm{SU}(6)$.

A compatible pair $\{J_\alpha, K\}$ $\implies$ SU(6) structure

Example: CY in IIA

H structure

$$J_+ = \tfrac{1}{2}\kappa(\Omega - \Omega^\sharp)$$
$$J_3 = \tfrac{1}{2}\kappa(I - \mathrm{vol}_6 - \mathrm{vol}_6^\sharp)$$

where $\kappa^2 = \mathrm{vol}_6 = \frac{\mathrm{i}}{8}\Omega \wedge \bar{\Omega}$ and I is complex structure.

V structure

$$K + \mathrm{i}\hat{K} = \mathrm{e}^{-\mathrm{i}\omega}$$

Example: CY in IIB

H structure

$$J_+ = \tfrac{1}{2}\kappa(\mathrm{e}^{-\mathrm{i}\omega} - \mathrm{e}^{-\mathrm{i}\omega^\sharp})$$
$$J_3 = \tfrac{1}{2}\kappa(\omega + \omega^\sharp - \mathrm{vol}_6 - \mathrm{vol}_6^\sharp)$$

where $\kappa^2 = \mathrm{vol}_6 = \frac{1}{6}\omega^3$.

V structure

$$K + \mathrm{i}\hat{K} = \Omega$$

Many examples

Minkowski

- Generalised Calabi–Yau (pure spinors)
- D3-branes on HK $\times \mathbb{R}^2$ in IIB
- Wrapped M5-branes on HK $\times \mathbb{R}^3$ in M-theory

AdS

- Sasaki–Einstein in 5d (IIB) and 7d (M-theory)
- Most general AdS_5 solutions in IIB and M-theory

Integrability

Differential conditions

Spaces of H and V structures admit action of generalised diffeomorphisms

$$\mathrm{GDiff} = \mathrm{Diff} \ltimes \mathrm{gauge}$$

and can define moment maps.

$$\text{integrability} \quad \Longleftrightarrow \quad \text{vanishing moment map}$$

Integrability for H structures

Consider space of H structures, coordinates $J_\alpha \in \mathcal{A}_{\mathsf{H}}$

- $\mathcal{A}_{\mathsf{H}}$ has hyper-Kähler metric, inherited fibrewise from

$$J_\alpha(x) \in W = \frac{\mathsf{E}_{7(7)} \times \mathbb{R}^+}{\mathsf{Spin}^*(12)}$$

  where W is HK cone over symmetric quaternionic-Kähler (Wolf) space.

- $\mathcal{A}_{\mathsf{H}}$ is also a HK cone, global $\mathbb{H}^+ = \mathsf{SU}(2) \times \mathbb{R}^+$.

Integrability for H structures

Hyper-Kähler structure on $\mathcal{A}_{\mathrm{H}}$ preserved by diffeos and gauge transformations, parametrised by $V \in \Gamma(E) \simeq \mathfrak{gdiff}$

$$\delta J_\alpha = L_V J_\alpha \quad \in T_J \mathcal{A}_{\mathrm{H}}$$

Moment maps

$$\mu_\alpha(V) = -\frac{1}{2}\epsilon_{\alpha\beta\gamma} \int_M \mathrm{tr}(J_\beta L_V J_\gamma)$$

where $\mu_\alpha \colon \mathcal{A}_{\mathrm{H}} \to \mathfrak{gdiff}^*$.

Integrability for H structures

Integrability

$$\mu_\alpha(V) = 0 \qquad \text{for all } V \in \Gamma(E)$$

For CY in IIA or IIB, gives $\mathrm{d}\Omega = 0$ or $\mathrm{d}\omega = 0$.

Moduli space

Structures related by GDiff are equivalent, moduli space is a hyper-Kähler quotient

$$\mathcal{M}_{\mathrm{H}} = \mathcal{A}_{\mathrm{H}} /\!\!/\!\!/ \mathrm{GDiff} = \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0) / \mathrm{GDiff}$$

HK cone over usual QK moduli space of hypermultiplets.

Integrability for V structures

Consider space of V structures, coordinates $K \in \mathcal{A}_{\mathsf{V}}$

- $\mathcal{A}_{\mathsf{H}}$ has special-Kähler metric, inherited fibrewise from

$$K \in P = \frac{\mathsf{E}_{7(7)} \times \mathbb{R}^+}{\mathsf{E}_{6(2)}}$$

where P is special-Kähler.

Moment maps

Again, $\mathfrak{gdiff}$ acts as $\delta K = L_V K \in T_K \mathcal{A}_{\mathsf{V}}$ and preserves SK structure, giving $\mu \colon \mathcal{A}_{\mathsf{V}} \to \mathfrak{gdiff}^*$

$$\mu(V) = \tfrac{1}{2} \int_M s(K, L_V K)$$

where $s(\cdot,\cdot)$ is $\mathsf{E}_{7(7)}$ symplectic invariant.

Integrability for V structures

Integrability

$$\mu(V) = 0 \qquad \text{for all } V \in \Gamma(E)$$

For CY in IIA or IIB, gives $(\mathrm{d}\Omega)_{3,1} = 0$ or $\omega \wedge \mathrm{d}\omega = 0$.

Moduli space

Structures related by GDiff are equivalent, moduli space is a symplectic quotient

$$\mathcal{M}_{\mathrm{V}} = \mathcal{A}_{\mathrm{V}} /\!\!/ \mathrm{GDiff} = \mu^{-1}(0)/\mathrm{GDiff}$$

SK cone over the usual SK moduli space of vector-multiplets.

Integrability for HV structures

$\mathcal{N} = 2$ backgrounds

H and V structures that are individually integrable are not sufficient. Need extra condition that couples them (unlike CY!)

$$\mu_\alpha(V) = \mu(V) = 0 \quad \text{plus} \quad L_K J_\alpha = L_{\hat{K}} J_\alpha = 0$$

For CY, recover $\mathrm{d}\omega = \mathrm{d}\Omega = 0$.

"Exceptional Calabi–Yau" $\Leftrightarrow$ $\mathcal{N} = 2$ with flux

Why these conditions?

Intrinsic torsion

SUSY equivalent to existence of generalised torsion-free connection D that is compatible with the structures

$$DJ_\alpha = DK = 0, \qquad T(D) = \{0\}$$

Integrability constrains same representations that appear in torsion.

[Coimbra, Strickland-Constable, Waldram]

Gauged $\mathcal{N} = 2$ supergravity

Rewrite 10d theory as $D = 4$, $\mathcal{N} = 2$ but keep all KK modes.

- Gauged 4d supergravity with infinite number of hypers and vectors.
- Integrability just $\mathcal{N} = 2$ vacuum conditions.

[Louis, Smyth, Triendl; Hristov, Looyestijn, Vandoren]

Outline

AdS_5 in type IIB

Sasaki–Einstein

$\mathcal{N} = 1$ AdS backgrounds in type IIB with $F_5 \neq 0$

$$ds^2_{10} = ds^2(\mathrm{AdS}_5) + ds^2(\mathrm{SE}_5), \qquad F_5 = dC_4 = 4\,\mathrm{vol}_5$$

Cone over SE_5 is Calabi–Yau.

- Contact form $\sigma = d\psi + \eta$, dual to Reeb vector ξ – nowhere-vanishing Killing vector.

$$\imath_\xi \sigma = 1, \qquad \imath_\xi \omega = \imath_\xi \Omega = 0, \qquad \omega \wedge \Omega = 0,$$
$$d\sigma = 2\omega, \qquad d\Omega = 3i\sigma \wedge \Omega, \qquad \mathcal{L}_\xi \Omega = 3i\Omega$$

$\mathrm{AdS}_5 \times \mathrm{SE}_5$ in type IIB

HV structure

$$\left.\begin{array}{ll} J_\alpha, & G = \mathrm{SU}^*(6) \\ K, & G = \mathrm{F}_{4(4)} \end{array}\right\} \quad G = \mathrm{USp}(6)$$

where $K \sim \mathrm{e}^{C_4}(\xi + \sigma \wedge \omega)$, $\quad J_+ \sim \mathrm{e}^{C_4}(\Omega + \Omega^\sharp)$.

Integrability for AdS

$$\mu_\alpha(V) = \lambda_\alpha \int_M c(K, K, V), \qquad L_K K = 0, \qquad L_K J_\alpha = \epsilon_{\alpha\beta\gamma}\lambda_\beta J_\gamma$$

where $c(\cdot,\cdot,\cdot)$ is $\mathrm{E}_{6(6)}$ cubic invariant.

- Implies K is "generalised Killing vector".

Marginal deformations

“Superpotential” deformations: $\delta J_\alpha = [R, J_\alpha] \neq 0, \quad \delta K = 0$

Dual to marginal deformations of $\mathcal{N} = 1$ SCFT

- R contains two-form and bivector components
- R generates flux

$$F_3 + \mathrm{i}H \propto f\sigma \wedge \bar{\Omega} + \ldots, \qquad \mathcal{L}_\xi f = 3\mathrm{i}f$$

where f is holomorphic on CY cone.

Marginal deformations

Which deformations can be extended to all orders? Higher-order calculations constrain f – long and difficult! [Aharony, Kol, Yankielowicz]

Moment map: no obstructions unless there are additional symmetries at $f = 0$.

Further quotient by

$$\mathfrak{g} = \{V \in \mathfrak{gdiff} : L_V J_\alpha = L_V K = 0\}$$

Supergravity version of [Green, Komargodski, Seiberg, Tachikawa, Wecht]

Marginal deformations

Example: S^5

Cone over S^5 is $\mathbb{C}^3$ with coordinates z_i: $\mathcal{L}_\xi z_i = \mathrm{i} z_i$

$$f = f^{ijk} z_i z_j z_k$$

SU(3) $\subset$ GDiff leaves $\{J_\alpha, K\}$ fixed, obstruction is SU(3) moment map

$$\gamma^i{}_j = f^{ikl}\bar{f}_{jkl} - \tfrac{1}{3}\delta^i{}_j f^{klm}\bar{f}_{klm} = 0$$

Only $10 - 8 = 2$ complex degrees of freedom in f.

AdS/CFT: $\gamma^i{}_j = 0$ just one-loop beta-function conditions, 2 exactly marginal deformations of $\mathcal{N} = 4$ SYM.

Summary

Summary

- Combines symplectic, complex, HK structures.
- Moduli space c.f. hyper-Kähler quotient.
- Reductions of type II and 11d supergravity to 4, 5 and 6 dimensions, Minkowski or AdS.
- AdS/CFT: deformations plus obstructions from symmetries.
  - Also works for 3d $\mathcal{N} = 2$ theories with M-theory duals.

Outlook

Questions

- $\mathcal{N} = 1$ backgrounds? Heterotic?
- Exponentiate deformations?
- Deformations are part of complex. Underlying DGLA?
- AdS: volume minimisation, calibrations?
- Links to topological string?