Discussion Session #1

Anthony Ashmore

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University of Chicago & Sorbonne Université

Summary of where we are

We have relatively fast and reliable methods to compute numerical

1. Ricci-flat metrics
2. Constant scalar curvature metrics
3. Extremal metrics
4. Hermitian Yang–Mills connections

Numerical methods

Ricci-flat metric on CY easiest as reduces to finding Kähler potential that minimises deviation from Ricci-flat

There are many approaches to this problem:

- Position space methods [Headrick, Wiseman '05; Doran et al. '07]
- Spectral methods [Donaldson '05; Douglas et al. '06; Braun et al. '07; Headrick, Nassar '09]
- Neural networks [Douglas et al. 20; Anderson et al. '20; Jejjala '20]

One can also try to find the metric directly (need to impose Kählerity, overlap conditions, etc.) [Anderson et al. '20; Jejjala '20]

Why do we want Ricci-flat metrics in physics?

String theory – solutions to vacuum Einstein equation

- Compactifications on CY threefolds with gauge connection – Ricci-flat + HYM
- Fix moduli or compute as function of them
- Effective physics controlled by zero-modes of Dirac operator – e.g. matter fields c.f. harmonic forms in $H^1(X, V_{\mathbf{R}})$

2d conformal field theories – CYs appear as target spaces

- Best understood for special points in moduli space, e.g. K3 as $\mathrm{T}^4/\mathbb{Z}_2$, or for protected quantities, e.g. counts of BPS states
- Spectrum of operators encoded in geometry

What would one do with a Ricci-flat metric?

Which questions are interesting but *not* answered by topology or algebraic geometry?

Spectrum of Laplacian [Donaldson '05; Braun et al. '08, AA '20]

Both captured by eigenmodes of $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$, i.e. (p,q)-eigenforms

$$\Delta\phi_n = \lambda_n\phi_n$$

where λ_n appear with multiplicity μ_n (c.f. finite symmetries)

Choose a finite basis of degree k_ϕ, (p,q)-forms $\{\alpha_A\}$ and expand as $\phi = \sum_A \phi_A\alpha_A$
– generalised eigenvalue problem for λ and ϕ_A

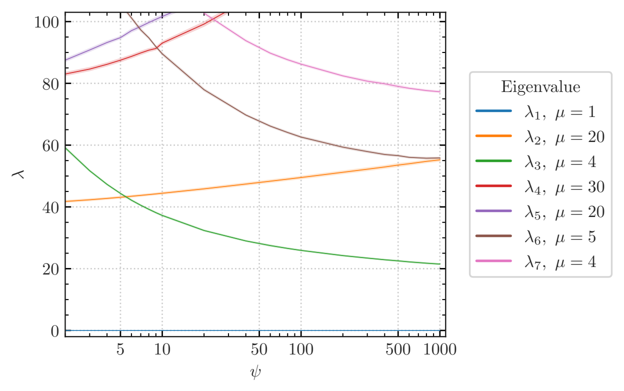
$$\begin{aligned} \langle\alpha_A, \Delta\alpha_B\rangle\langle\alpha_B, \tilde{\phi}\rangle &= \lambda\langle\alpha_A, \alpha_B\rangle\langle\alpha_B, \tilde{\phi}\rangle \\ \Rightarrow \quad \Delta_{AB}\tilde{\phi}_B &= \lambda O_{AB}\tilde{\phi}_B \end{aligned}$$

Scalar spectrum for large complex structure [AA, Ruehle ‘21]

Dwork family of quintics Q_ψ in $\mathbb{P}^4$ with

$$Q_\psi(z) \equiv z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi\, z_0 z_1 z_2 z_3 z_4 = 0$$

Look at scalar spectrum with varying complex structure $\psi \in [2, 1000]$



- Eigenvalues have multiplicity due to $(S_5 \times \mathbb{Z}_2) \ltimes (\mathbb{Z}_5)^3$ symmetry
- Lowest-lying eigenvalue bounds diameter
- SYZ: T^3 fibers shrink to zero size, spectrum reduces to that on S^3

“Typical” properties

What are the “typical” properties of a given Calabi–Yau averaged over its moduli?

- Spectral gap?
- Statistics of spectrum of Δ?
- Statistics of Yukawa couplings? etc.

Important for landscape of string compactifications and CFTs

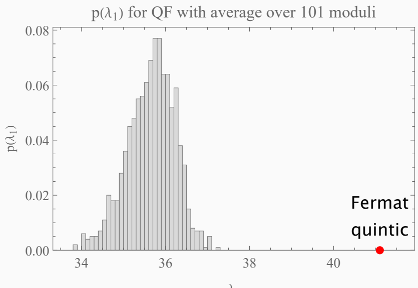
- Without fixing moduli, what kind of physics is possible?
- Averaged quantities often have nicer behaviour than any single instance, e.g. random matrix theory, ensemble-averaged CFTs

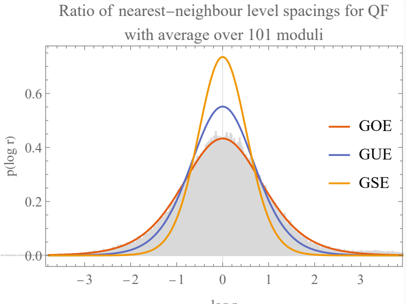
Random quintic threefold given by choosing c_{mnpqr} from unit disk in $\mathbb{C}$

$$Q \equiv \sum_{m,n,p,q,r} c_{mnpqr} z_m z_n z_p z_q z_r = 0$$

Compute a numerical CY metric and the scalar spectrum for 1000 instances

- Statistics, e.g. level spacings $r_i = \frac{\lambda_{i+1}-\lambda_i}{\lambda_i-\lambda_{i-1}}$, match Gaussian Orthogonal Ensemble





What would one do with a Calabi–Yau metric?

Can study modes in CY background – c.f. Laplacian

Non-BPS checks of mirror symmetry / SYZ? Fiber/base decomposition for elliptic or K3 fibrations?

Study geodesics on CY manifolds? Conjecture that number of locally length minimising closed geodesics grows as L^D [Gao, Douglas '13]

Is an explicit expression more useful? Progress for K3 metrics [Kachru et al. '18, '20; Gaiotto et al. '09]

Advances in numerical methods

Is it more important to have *rigorous* approximation schemes or *fast* ones?

Is it currently worth trying to combine metrics, connections, moduli space metrics, etc?

$\mathrm{SU}(3)$ structure metrics? (See Lara's talk) G_2 metrics?

Can one learn moduli dependence in high-dimensional moduli spaces?

What developments needed for neural networks to become general PDE solvers?

Mathematics vs numerics

Experimental mathematics – could numerics suggest research directions?

- Ricci-flat metrics with generic holonomy on compact, simply connected manifolds? What about stable metrics? [Acharya '20] (See also Nicos' talk)
- e.g. does $S^2 \times S^2$ admit a Ricci-flat metric? Can neural networks help?

Examples where we do *not* trust the numerical solutions without proof of existence?

- Existence of non-Kähler metrics? G_2 metrics?
- Hull–Strominger system – small cycles in geometry $\Rightarrow$ any physics interpretation? [Melnikov et al. '14; Lotay, Sá Earp '21]