Exactly marginal deformations and their supergravity duals

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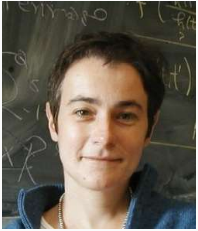

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Motivation

Focus on 4d $\mathcal{N}=1$ SCFTs with type IIB duals

- Canonical example

$$\text{IIB on } \mathrm{AdS}_5 \times S^5 \quad \Leftrightarrow \quad \mathcal{N}=4 \text{ SYM}$$

- Generalisation with all fluxes

$$\text{IIB on } \mathrm{AdS}_5 \times M \quad \Leftrightarrow \quad \mathcal{N}=1 \text{ SCFT}$$

Known solutions

- e.g. metric + $F_5 \Rightarrow M$ is Sasaki–Einstein
- e.g. Pilch–Warner, β deformation [Lunin, Maldacena '05]

4d $N = 4$ SYM in $N = 1$ language

Three chiral fields Φ^i with SU(3) flavour symmetry and superpotential

$$\mathcal{W} = \epsilon_{ijk} \,\mathrm{tr}(\Phi^i \Phi^j \Phi^k)$$

F-term conditions imply Φ^i commute: $\partial_1 \mathcal{W} \propto [\Phi^2, \Phi^3] = 0$, etc.

Chiral ring $\leftrightarrow$ ring of holomorphic functions on $C(S^5) = \mathbb{C}^3$:

$$\mathcal{O}_f = f_{i_1 \dots i_n} \,\mathrm{tr}(\Phi^{i_1} \dots \Phi^{i_n}) \quad \leftrightarrow \quad f(z^i)$$

Hilbert series: graded count of single-trace mesonic operators

$$H(t) = \sum_k n_k t^k = \frac{1}{(1-t)^3} = 1 + 3t + 6t^2 + 10t^3 + \dots$$

Marginal deformations

e.g. $\mathcal{N}=1$ deformations of $\mathcal{N}=4$ SYM [Leigh, Strassler '95]

$$\mathcal{W} = f_{ijk}\,\mathrm{tr}(\Phi^i\Phi^j\Phi^k)$$

- $f_{ijk} \in \mathbf{10}_{\mathbb{C}}$ of SU(3) – 10 complex d.o.f.
- One-loop beta functions

$$f_{ikl}\bar{f}^{jkl} - \tfrac{1}{3}\delta_i^j f_{klm}\bar{f}^{klm} = 0$$

Two exactly marginal couplings form conformal manifold [Kol '02, Kol '10, Green et al. '10]

$$\mathcal{M}_c = \{f_{ijk}\} \,//\, \mathrm{SU}(3) = \{f_{ijk}\} \,/\, \mathrm{SL}(3,\mathbb{C})$$

Superpotential and chirals

At the $\mathcal{N} = 4$ point, we can choose

$$\Delta\mathcal{W} = f_\beta \,\mathrm{tr}(\Phi^1\Phi^2\Phi^3) + f_\lambda \,\mathrm{tr}\left[(\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3\right]$$

F-term relations define non-commutative Sklyanin algebra [Ginzburg '06]

Chiral operators for generic f_β and f_λ counted by [Van den Bergh '94]

$$H(t) = \frac{(1+t)^3}{1-t^3} = 1 + 3t + 3t^2 + 2t^3 + \dots$$

- Two marginal deformations again
- Not known for generic $\mathcal{N} = 1$ SCFTs

Dual geometries?

Can we understand the dual geometries?

- $f_\lambda = 0$: "β deformation", preserves $U(1)^2$ isometry, exact dual solution known [Lunin, Maldacena '05]
- Generic: no isometries (other than $U(1)_R$)
- For S^5, *tour de force* 3rd-order perturbative analysis [Aharony, Kol, Yankielowicz '02], but full solution not known

Can we count the chiral operators?

Goals of talk

1. Describe supergravity analogue of holomorphic data encoded by $\mathcal{W}$
2. Show how holomorphic data determines solution up to action of complexified diffeos + gauge
3. Compute Hilbert series for deformed SCFTs from dual geometry

AdS_5 in type IIB & generalised geometry

Supersymmetric AdS_5 backgrounds

Generic type IIB solution preserving 8 supercharges with fields $(\Delta, \tau, H_3, F_3, F_5, g)$

$$\mathrm{d}s^2_{10} = e^{2\Delta}\mathrm{d}s^2(\mathrm{AdS}_5) + \mathrm{d}s^2(M)$$

Symmetries: GDiff $\sim$ diffeos + p-form gauge

$$\delta B^i = \mathrm{d}\lambda^i, \qquad \delta C_4 = \mathrm{d}\rho - \tfrac{1}{2}\epsilon_{ij}\mathrm{d}\lambda^i \wedge \mathrm{d}B^j$$

Supersymmetry: fermions $= 0$ and $\delta_\epsilon(\text{fermions}) = 0$

$$\nabla_m \epsilon + (\text{flux})_m \cdot \epsilon = 0, \qquad \gamma^m \nabla_m \epsilon + \text{flux} \cdot \epsilon = 0$$

with ϵ stablised by USp(6) [Coimbra, Strickland-Constable, Waldram '14]

Example: Sasaki–Einstein

e.g. M is Sasaki–Einstein

Geometry defined by nowhere-vanishing tensors σ, j and Ω

- Defined by spinor bilinears: $j_{mn} \sim \bar{\epsilon}\gamma_{mn}\epsilon$, etc.
- $\xi = g^{-1}\sigma$ defines $\mathrm{U}(1)_R$ of dual SCFT

Tensors satisfy algebraic conditions

$$\imath_\xi \sigma = 1, \qquad \imath_\xi j = \imath_\xi \Omega = j \wedge \Omega = 0, \qquad j^2 = \tfrac{1}{2}|\Omega|^2$$

Invariant under $\mathrm{SU}(2) \subset \mathrm{GL}(5, \mathbb{R})$

Example: Sasaki–Einstein

Supersymmetry implies differential conditions on invariant tensors

$$
\mathrm{d}\sigma = 2j, \qquad \mathrm{d}\Omega = 3\mathrm{i}\sigma \wedge \Omega,
$$
$$
F_5 = 4(\mathrm{vol}(\mathrm{AdS}_5) + \mathrm{vol}(M_5))
$$

- ξ is Killing vector
- Corresponds to SU(2) structure with singlet intrinsic torsion

SUSY backgrounds with flux

Long history of using G-structures and generalised geometry to analyse supersymmetric flux backgrounds

Generic AdS_5 case: spinor ϵ defines exceptional Sasaki–Einstein structure, stabilised by USp(6) [AA, Petrini, Waldram '16]

- Defined by pair (X, K)

$$X \sim \text{hyper d.o.f.} \qquad K \sim \text{vector d.o.f.}$$

  Tensors in $E_{6(6)} \times \mathbb{R}^+$ generalised geometry [Hull '07, Pacheco, Waldram '08]

- Construct tensors as irreps of $E_{6(6)} \times \mathbb{R}^+$

$$\mathrm{GL}(5, \mathbb{R}) \to E_{6(6)} \times \mathbb{R}^+$$

Generalised structures

Spinor ϵ defines the pair (X, K)

$$
\begin{array}{ccccc}
 & & E_{6(6)} \times \mathbb{R}^+ & & \\
 & \swarrow & & \searrow & \\
(K \in E) \quad F_{4(4)} & & & & SU^*(6) \quad (X \in \det T^* \otimes \mathrm{ad}_{\mathbb{C}}) \\
 & \searrow & & \swarrow & \\
 & & USp(6) & & \\
 & & (X, K) & &
\end{array}
$$

Intersect on USp(6) if compatible

$$
X \cdot K = 0, \qquad \mathrm{tr}(X\bar{X}) = c(K, K, K)^2
$$

(X, K) equivalent to specifying all supergravity fields for solution

Example: Sasaki–Einstein

Recall structure defined by (σ, j, Ω)

K structure defines "contact structure"

$$K = \xi - \sigma \wedge j \quad \in T \oplus \Lambda^3 T^* \subset E$$

X structure defines "Cauchy–Riemann structure"

$$X = \mathrm{e}^{\frac{1}{2}\mathrm{i} j^2} u^i \, \sigma \wedge \Omega \quad \in 2\Lambda^3 T^* \subset \mathrm{ad}_{\mathbb{C}} \otimes \det T^*$$

with $u^i = \tau_2^{-1/2}(\tau, 1)^i$ and $\tau = \chi + \mathrm{i}\mathrm{e}^{-2\phi}$

Supersymmetry

Symmetries act by a generalised Lie derivative along generalised vector $V = v^a + \lambda^i_a + \rho_{abc} + \sigma^i_{abcde}$:

$$L_V = \mathcal{L}_v - (\mathrm{d}\lambda^i + \mathrm{d}\rho)$$
$$\sim \text{GDiff} = \text{diffeo} + \text{gauge}$$

Supersymmetry of the solution is then equivalent to [AA, Petrini, Waldram '16]

$$L_K K = 0, \qquad L_K X = 3\mathrm{i}X,$$
$$\mu_+(V) = 0, \qquad \mu_3(V) = \int_M c(K, K, V) \qquad \forall\, V$$

- Equivalent to supersymmetry conditions derived in [Gauntlett et al. '04]
- $\frac{2}{3} L_K$ generates $U(1)_R$ of dual SCFT

Holomorphic data & counting chirals

Deformed solutions

Can we solve for the general supergravity solution dual to the deformed field theories? *Unlikely!*

- Solving for generic solution seems intractable – no isometries, harder than Monge–Ampère for Calabi–Yau

Instead, focus on holomorphic data

$$\mu_+(V) = 0, \qquad L_K K = 0, \qquad L_K X = 3\mathrm{i}X$$

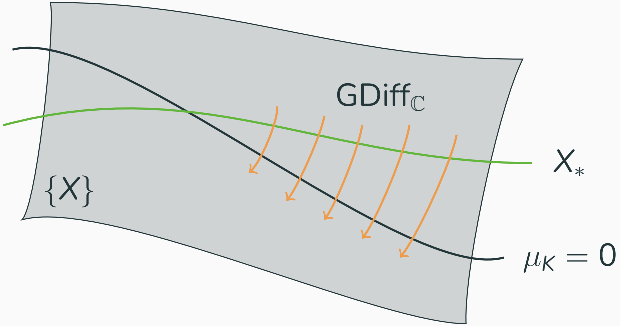
- "Exceptional Sasaki"
- Space of X that solve these conditions is still Kähler
- Moduli space given by μ_3 moment map + GDiff quotient – equivalent to $\mathrm{GDiff}_{\mathbb{C}}$ quotient

General argument

Given deformed solution (X_*, K) to holomorphic conditions, can show that full solution exists:

1. $\mu_K(V) = \mu_3(V) - \int_M c(K, K, V)$ is moment map for GDiff with fixed K
2. (X_*, K) matches exactly marginal solutions for infinitesimal deformations
3. Open subset of stable points that lie on orbits of $\mathrm{GDiff}^K_{\mathbb{C}}$ will intersect $\mu_K = 0$ – all (X_*, K) are stable and thus can be mapped to full solutions
4. Different X_* flow to different solutions unless there are isometries, in agreement with field theory [Kol '02, Kol '10, Green et al. '10]

Physical interpretation



1. Fixing an orbit $[X] \simeq \mathrm{GDiff}_{\mathbb{C}} \cdot X$ fixes the superpotential $\mathcal{W}$ of dual SCFT
2. $L_K X = 3iX$ fixes $\Delta = 3$ – marginal deformation
3. Motion along orbit $\equiv$ renormalisation of Kähler potential

Example: S^5 again

Mesonic operators $\mathrm{tr}(\Phi \dots)$ $\quad\leftrightarrow\quad$ holomorphic functions $f(z)$ on cone

- Marginal $\Rightarrow \mathcal{L}_\xi f = 3\mathrm{i} f$

Cone is $C(S^5) = \mathbb{C}^3$; functions are $f = f_{ijk} z^i z^j z^k$

Recall, at S^5 point

$$X = \mathrm{e}^{\frac{1}{2}\mathrm{i} j^2} u^i \, \sigma \wedge \Omega \sim u^i \, \sigma \wedge \Omega \quad \text{up to } \mathrm{GDiff}_{\mathbb{C}}$$

How do we deform this by f?

X_* for deformed S^5 background

New exact family of solutions to holomorphic conditions

$$K = \xi - \sigma \wedge j, \qquad X_*(f) = e^{b^i(\tau,f)}(df + v^i(\tau,f)\,\sigma \wedge \Omega)$$

with $b^i \in \Lambda^2 T^*_{\mathbb{C}}$ linear in f and v^i quadratic

- In S^5 case and f cubic, reproduces leading parts of [Aharony, Kol, Yankielowicz '02]
- For $f = z^1 z^2 z^3$, can solve for explicit $\mathrm{GDiff}_{\mathbb{C}}$ to take solution to exact β-deformed solution
- Works for deformation of any Sasaki–Einstein background – $T^{1,1}$, etc.

Chiral spectrum

What can we calculate using this (partial) solution?

X_* fixes superpotential so should encode space of mesonic operators

$$\mathcal{O}_f = f_{ijkl\dots}\,\mathrm{tr}(\Phi^i\Phi^j\Phi^k\Phi^l\dots)$$

Can count these graded by R-charge $\to$ Hilbert series

- Counting for Sasaki–Einstein point done by [Eager, Schmude, Tachikawa '12]
- But we want to count for the deformed theory!

Chiral spectrum

Counting δX up to $\mathrm{GDiff}_{\mathbb{C}}$ defines a cohomology which counts chiral operators

$$\text{chirals} \sim \frac{\{\delta X \,|\, \delta\mu_+ = 0\}}{\{\delta X = L_V X\}}$$

- Drop $L_K X = 3\mathrm{i}X$ condition to count tower of KK modes
- Counting depends only on class $[X] = [X_*]$

Calculating the cohomoloy

Easiest when the deformed solution is generic – $\mathrm{d}f \neq 0$

Cohomology then reduces to [Tasker '21]

$$\dots \xrightarrow{\mathrm{d}} \mathrm{d}f \wedge \Lambda^p T^*_{\mathbb{C}} \xrightarrow{\mathrm{d}} \mathrm{d}f \wedge \Lambda^{p+1} T^*_{\mathbb{C}} \xrightarrow{\mathrm{d}} \dots$$

which can be computed using Kohn–Rossi cohomology of original Sasaki–Einstein

Counting chirals

Hilbert series

$$H(t) \equiv \sum_k n_k t^k = 1 + \mathcal{I}_{\text{s.t.}}(t) - [k \equiv_3 0, k > 0] t^{2k}$$

e.g. deformed S^5 dual to deformed $\mathcal{N} = 4$ SYM with

$$f = f_\beta z^1 z^2 z^3 + f_\lambda \left[(z^1)^3 + (z^2)^3 + (z^3)^3 \right]$$

Hilbert series is

$$H(t) = \frac{(1+t)^3}{1-t^3} = 1 + 3t + 3t^2 + 2t^3 + \dots$$

in agreement with [Van den Bergh '94]

New results

e.g. $T^{1,1}$ – undeformed result

$$H(t) = \frac{1 + t^{3/2}}{(1 - t^{3/2})^3} = 1 + 4t^{3/2} + 9t^3 + 16t^{9/2} \ldots$$

For Klebanov–Witten theory with generic deformed superpotential

$$H(t) = \frac{1 + 4t^{3/2} + 2t^3}{1 - t^3} = 1 + 4t^{3/2} + 3t^3 + 4t^{9/2} + \ldots$$

- Matches explicit counting of chiral fields modulo F-term relations up to $k = 21/2$ [Tasker '21]
- No previous calculation of cyclic homology / chirals for deformed theory!

New results for $\#n(S^2 \times S^3)$, etc.

Summary

Background geometry naturally encodes superpotential of dual SCFT

Can find supergravity solution for deformations up to $\mathrm{GDiff}_{\mathbb{C}}$ action

Class of structure $[X]$ determines spectrum of chiral operators

Future

- Same/similar formalism for $\mathrm{AdS}_5/\mathrm{AdS}_4$ in M-theory
- Cohomology gives supersymmetric index
- a-maximisation for generic supersymmetric backgrounds – $a^{-1} \sim \int_M c(K, K, K)$

Backup slides

K structure

Generalised vector V^A parametrises diffeos + gauge transformations

$$27 \sim E \simeq T \oplus 2T^* \oplus \Lambda^3 T^* \oplus 2\Lambda^5 T^*$$
$$V^A = v^a + \lambda^i_a + \rho_{abc} + \sigma^i_{abcde}$$

Invariant cubic form on E

$$c(V, V, V) = -\tfrac{1}{2}\imath_v \rho \wedge \rho + \cdots \in \det T^*$$

K structure defined by

$$K \in E \quad \text{s.t.} \quad c(K, K, K) > 0$$

- Generalised vector invariant under $F_{4(4)} \in E_{6(6)}$

X structure

e.g. adjont elements

$$78 \sim \mathrm{ad} \simeq 3\mathbb{R} \oplus (T \otimes T^*) \oplus 2\Lambda^2 T^* \oplus 2\Lambda^2 T \oplus \Lambda^4 T^* \oplus \Lambda^4 T$$
$$R^A{}_B = \cdots + B^i{}_{ab} + \cdots + C_{abcd} + \ldots$$

X structure defined by

$$X \in \mathrm{ad}_{\mathbb{C}} \otimes \det T^* \quad \text{s.t.} \quad \mathrm{tr}(X\bar{X}) \neq 0$$

- Complex adjoint tensor invariant under $\mathrm{SU}^*(6) \in \mathrm{E}_{6(6)}$
- $X = \kappa(J_1 + \mathrm{i}J_2) = \kappa J_+$ defines $\mathfrak{su}_2$ triplet

$$[J_\alpha, J_\beta] = 2\kappa\epsilon_{\alpha\beta\gamma}J_\gamma, \qquad \mathrm{tr}(J_\alpha J_\beta) = -\kappa^2\delta_{\alpha\beta}, \qquad \kappa^2 \in \det T^*$$

Moment maps

The μ_α are a triplet of moment maps for the action of

$$\mathrm{GDiff} \simeq \mathrm{diffeo} + \mathrm{gauge}$$

Infinitesimally, $V \in \Gamma(E) \simeq \mathfrak{gdiff}$ acts by

$$\delta J_\alpha = L_V J_\alpha$$

Action preserves hyper-Kähler structure on space of J_α so that

$$\mu_\alpha(V) = -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma} \int_M \mathrm{tr}(J_\beta L_V J_\gamma)$$

Marginal vs exactly marginal deformations

The field theory result of [Kol '02, Kol '10, Green et al. '10] that all marginal deformations are exactly marginal unless there is a global symmetry follows directly from moment map structure

e.g. $\mathrm{AdS}_5 \times \mathrm{S}^5$, (X, K) preserved by SU(3)

- Linearised deformation parameterised by $f = f_{ijk} z^i z^j z^k$
- $\mu_\alpha(V)$ trivially zero for $V \in \mathrm{SU}(3)$
- Further moment map for SU(3) and quotient on $\{f_{ijk}\}$

$$\mu_{\mathrm{SU}(3)} \equiv f_{ikl}\bar{f}^{jkl} - \tfrac{1}{3}\delta_i^j f_{klm}\bar{f}^{klm} = 0$$

gives space of exactly marginal couplings